Proceedings of the DIS'2004, Štrbské Pleso, Slovakia

THE QCD S–MATRIX IN THE HIGH–ENERGY LIMIT

EDMOND IANCU
Service de Physique Théorique, CEA/DSM/SPhT,
91191 Gif-sur-Yvette Cedex, France
E-mail: eiancu@cea.fr

I discuss elastic dipole–dipole scattering in QCD at high energies, with emphasis on the relation between Mueller's dipole picture and the Color Glass Condensate, and on the importance of rare fluctuations for the high–energy limit of the S–matrix.

1 Dipole–dipole scattering: Color Dipoles versus Color Glass

Let me consider the simplest scattering process that one can think of in perturbative QCD — the elastic scattering between two small color dipoles (or 'onia') at zero impact parameter, in the center–of–mass (COM) frame and at relative rapidity Y — and address a basic question: What is the high energy limit of the S–matrix for this collision ? On physical grounds, the answer seems quite clear: $S_Y \to 0$ as $Y \to \infty$ (indeed, at sufficiently high energies, the wavefunctions of the two dipoles contain so many gluons that the probability S_Y^2 that no interaction take place in the collision is very small, $S_Y^2 \ll 1$, and it vanishes when $Y \to \infty$). But deriving even this simple limit from perturbative QCD turns out to be quite non–trivial, as demonstrated, for instance, by the 'small–x problem' of the linear evolution equations: The BFKL equation [1], which is supposed to resum the dominant radiative corrections at high energies, predicts a scattering amplitude $T_Y \equiv 1 - S_Y$ which rises exponentially with Y, thus eventually violating the unitarity bound $T_Y \leq 1$.

Over the last two decades, several formalisms have been gradually developed which go beyond the BFKL evolution by including those effects which restore unitarity at high energies. These effects can be generically characterized as 'multiple scatterings', but their precise interpretation, and also their mathematical description, depend crucially upon the choice of a frame: Whereas in the COM frame, unitarity corrections start to manifest themselves, when increasing Y, as genuine multiple scatterings ("multiple pomeron exchanges", or "pomeron loops") [2], in an asymmetric frame in which one of the two dipoles is much faster than the other one, and thus carries most of the evolution, unitarity also appears as non–linear effects in the gluon distribution of the evolved dipole, or "gluon saturation" [3,4].

This can be best appreciated on a simple example: In the single–pomeron–exchange, or BFKL, approximation (valid so long as $T_Y \ll 1$) and in the COM frame, the scattering amplitude for two identical dipoles of transverse size r_0 can be estimated as [2]:

$$T(r_0, r_0, Y) \sim \alpha_s^2 \, n^2(r_0, r_0, Y/2), \tag{1}$$

where $n(r_0, r_0, Y/2)$ is the number density of radiated dipoles of size r_0 in the wavefunction of the parent dipole of size r_0 and with rapidity $Y/2$, and α_s^2 is the scattering amplitude for two elementary dipoles of similar sizes. The BFKL evolution is encoded in the dipole number density, which grows roughly like $n(r_0, r_0, Y) \sim \mathrm{e}^{\omega_{\mathbb{P}} Y}$ with $\omega_{\mathbb{P}} = (4 \ln 2)\alpha_s N_c/\pi$ (the 'BFKL intercept'). Unitarity corrections become important when $T_Y \sim 1$, that is, for $Y = Y_0$ with

$$Y_0 \simeq \frac{1}{\omega_{\mathbb{P}}} \ln \frac{1}{\alpha_s^2}. \tag{2}$$

For $Y \sim Y_0$, and in this particular frame, BFKL fails to describe correctly the scattering amplitude, but it remains a reasonable approximation for the *wavefunctions* of the evolved dipoles. Indeed, the non–linear effects within each wavefunction are rather measured by $\alpha_s^2 n(r_0, r_0, Y/2)$ — the scattering amplitude between a given dipole and the other dipoles within the same wavefunction —, which is still small, of $\mathcal{O}(\alpha_s)$, when $Y \sim Y_0$. Therefore, saturation effects in the COM frame start to manifest themselves only at higher rapidities $Y \gtrsim 2Y_0$.

But when the same scattering is viewed in the asymmetric frame in which one of the dipoles is nearly at rest, then $T(r_0, r_0, Y) \sim \alpha_s^2\, n(r_0, r_0, Y)$, with $n(r_0, r_0, Y)$ referring to the evolved dipole. Again, T becomes of $\mathcal{O}(1)$ for $Y \sim Y_0$, but when this happens, the non–linear effects in the wavefunction of the evolved dipole are also of $\mathcal{O}(1)$, and must be treated on the same footing as the multiple scattering.

The language of the example above is that of the *Color Dipole Picture* (CDP), a formalism originally developed by Mueller [2] which cannot accomodate saturation effects — the onium wavefunction is rather described by BFKL (together with a large–N_c approximation), as a collection of dipoles which evolves through dipole splitting —, but which can describe unitarity corrections in the COM frame, as the multiple scattering between several pairs of dipoles from the two incoming 'onia'. This formalism has been numerically implemented by Salam [5]. It also lies at the basis of Kovchegov's derivation of a non–linear evolution equation for the scattering between a dipole and a large nucleus [6], to which I shall return later.

More recently, a different formalism has been developed, the *Color Glass Condensate* (CGC) [4,7], which is specially tailored to describe saturation in the wavefunction of an energetic hadron. The CGC is the matter made of small–x gluons in the high–density environment characteristic of saturation. It is characterized by a *saturation plateau* at relatively low transverse momenta : the gluon modes with $k_\perp \leq Q_s(Y)$ have large occupation numbers, of $\mathcal{O}(1/\alpha_s N_c)$, but which increase only slowly when increasing y or decreasing $k_\perp$. The *saturation momentum* $Q_s(Y)$ [3] is an intrinsic scale generated by the non–linear dynamics; it increases rapidly with Y, so at large Y it provides a *hard* scale for the running of the coupling.

The mathematical language of the CGC theory is that of *classical statistical physics* : This is a theory for classical color fields endowed with a (functional) probability distribution which evolves with Y according to a (functional) Fokker–Planck equation — the JIMWLK equation [7,8,9] — in which Y plays the role of 'time'. The classical fields are generated by color sources (the parent dipole plus radiated gluons) at rapidities larger than the rapidity Y of interest, whose dynamics is therefore 'frozen' by Lorentz time dilation. When increasing Y in one more step ($Y \to Y + dY$ with $\alpha_s dY \sim 1$), a new layer is added to the classical fields in longitudinal direction, corresponding to 'integrating out' the emitted gluons with rapidities between Y and $Y + dY$. The evolution thus generates a *random walk* in the configuration space of color fields, in which the elementary step consists in the emission of a small–x gluon in the background of the color fields created in the previous steps. In this description, saturation effects enter through the fact that the probability for induced gluon radiation is non–linear in the background field and cannot exceed one. The BFKL evolution is recovered in the limit where the background field is weak, corresponding to low gluon occupation numbers.

Computing a scattering cross–section in the CGC formalism is a delicate task: Since the color glass is characterized by strong fields, the standard factorization schemes for high–energy scattering are not bound to apply, and in general we only know how to describe the collision between a CGC and a simple projectile, like a (bare) dipole. Specifically, the S–matrix for elastic CGC–dipole scattering is computed in the eikonal approximation as ($x_\perp$ and $y_\perp$ are the transverse coordinates of the quark and the antiquark which make up the dipole)

$$S_Y(x_\perp, y_\perp) = \int \mathrm{D}[\alpha]\; W_Y[\alpha]\; \frac{1}{N_c}\, \mathrm{tr}\big(U^\dagger(x_\perp) U(y_\perp)\big), \tag{3}$$

where $U^\dagger(x_\perp)$ and $U(y_\perp)$ are Wilson lines describing the color precession of the quark, or the antiquark, in the color field $A_a^+ \equiv \alpha_a$ of the CGC, and $W_Y[\alpha]$ is the probability distribution for this field, which obeys the JIMWLK equation alluded to above. By using the latter, one can derive an evolution equation for S_Y; however, this is not a closed equation, but only the first equation in an infinite hierarchy originally obtained by Balitsky [10]. Still, a closed equation for S_Y can be obtained within a mean field approximation (MFA): This is the non–linear equation originally derived by Kovchegov [6] (in a different physical context, though, namely for the scattering between a dipole and a large nucleus, where the MFA is better under control), and which is generally dubbed as the Balitsky–Kovchegov (BK) equation.

Returning to our original problem of the dipole–dipole scattering, one sees that, within the CGC formalism, this scattering is most simply described in an *asymmetric* frame, in which one of the dipoles carries most of the total rapidity and has evolved into a CGC, while the other dipole is rather slow and can be described as a bare $q\bar{q}$ pair, without additional gluons. This feature complicates the comparison with the CDP formalism, where the same problem is most naturally formulated in the COM frame. Still, as shown recently [11] through analytic manipulations, the two formalisms *are* in fact equivalent, within the range in Y in which they are both supposed to apply. This equivalence is the first point that I would like to slightly elaborate on in what follow. The other point is the role of *rare fluctuations* in the approach of the S–matrix towards the 'black–body' limit $S = 0$ [12]. (See also Ref. [13] for a related analysis.)

2 COM scattering between two color glasses

The interesting rapidity range for comparing CDP to CGC is $Y_0 \lesssim Y < 2Y_0$ (cf. Eq. (2)) : for $Y \ll Y_0$, both formalisms reduce to the BFKL approximation (and thus are obviously equivalent), while for $Y \geq 2Y_0$, CDP fails to apply because of saturation effects in the wavefunctions of the incoming dipoles (in COM frame). The first step towards establishing the equivalence consists in factorizing the S–

matrix for the elastic scattering between two color glasses in the COM frame. The second step consists in showing that the JIMWLK evolution of the wavefunction of a color dipole reduces to the corresponding BFKL evolution (as implemented in CDP) in the weak field approximation and for large N_c.

Concerning the first step, the factorization proposed in [11] reads as follows:

$$S_Y \,=\, \int \mathrm{D}[\alpha_R]\; W_{Y/2}[\alpha_R] \int \mathrm{D}[\alpha_L]\; W_{Y/2}[\alpha_L]\; \mathrm{e}^{\,\mathrm{i}\int d^2z_\perp\, \nabla^i\alpha_L^a(z_\perp)\nabla^i\alpha_R^a(z_\perp)}\,, \qquad (4)$$

where the symbols L and R stand for the left–mover and the right–mover, respectively. The exponential factor is recognized as the coupling between the color charge density in one system (e.g., $\rho_L^a = -\nabla^2\alpha_L^a$) and the color field in the other system. It describes multiple (eikonal) scattering in the approximation that the *individual* color sources within each system undergo at most *single* scattering. (But *global* multiple scattering is still allowed, as the simultaneous scattering of several constituents from the two systems.) Eq. (4) is correct for $Y_0 \lesssim Y < 2Y_0$ since in that range none of the two incoming color glasses is at saturation, and multiple scattering of a *single* gluon is indeed negligible. However, *global* multiple scattering *is* important, because each system involves a large number of constituents.

As for the second step, one needs to show that the CGC and CDP descriptions of an evolved dipole (the onium) become equivalent with each other when the CGC formalism is simplified by using the weak field (or BFKL) approximation together with the large–N_c limit. To that aim, we have shown in [11] that (i) the parent dipole can be represented as a color glass, and (ii) its evolution with Y, as described by the correspondingly simplified version of the JIMWLK equation, can be reformulated as the evolution of a system of dipoles, in agreement with CDP.

The manipulations in Ref. [11] imply that Eq. (4) can be rewritten in the form expected in CDP [2], namely, as the S–matrix for the scattering between two systems of dipoles. Schematically,

$$S_Y = \sum_{N,N'=1}^{\infty} \int d\Gamma_N P_N(Y/2) \int d\Gamma_{N'} P_{N'}(Y/2) \exp\Big\{-\sum_{i=1}^{N}\sum_{j=1}^{N'} T_0(i|j)\Big\}\,, \qquad (5)$$

where $P_N(Y/2)$ is the probability density for producing a given configuration of N dipoles after a rapidity evolution $Y/2$ (this depends upon the transverse coordinates of the dipoles and evolves through dipole splitting according to the BFKL kernel [2,11]), the integral $d\Gamma_N$ runs over the dipole coordinates, and $T_0(i|j)$ is the elementary scattering amplitude (via two gluon exchange) between the dipole i in the first onium and the dipole j in the second one.

Note that, although the exponential in Eq. (5) looks formally like a Glauber approximation (the multiple scattering series is resummed as the exponential of minus the amplitude for a single scattering), this exponentiation holds only *configuration by configuration*. After averaging over all such configurations, the resulting S–matrix differs significantly from the simple exponential of the one pomeron exchange[a]. This difference is particularly pronounced in the high energy regime at

[a] The one–pomeron–exchange ampolitude in Eq. (1) is recovered from Eq. (5) as the linear term in the expansion of the exponential there.

$Y \gtrsim Y_0$, where S_Y is very small: The naive exponentiation of the one pomeron exchange in Eq. (1) would predict (κ_0 is an unknown factor):

$$S_Y \sim \exp\Big\{ -\kappa_0 \alpha_s^2 n^2(r_0, r_0, Y/2)\Big\} \quad \text{with} \quad n^2(Y/2) \sim \mathrm{e}^{\omega_{\mathbb{P}} Y}\,, \tag{6}$$

whereas the Monte–Carlo calculation of Eq. (5) by Salam rather yields [5] :

$$S_Y \sim \mathrm{e}^{-\kappa \bar\alpha_s^2 Y^2} \quad \text{with} \quad \bar\alpha_s = \alpha_s N_c/\pi, \qquad \kappa \approx 0.72\,, \tag{7}$$

which, although small, is considerably larger than the naive estimate (6). Understanding this difference brings me to my next point, namely:

3 On the importance of rare fluctuations at high energies

Why is the S–matrix in Eq. (7) approaching the black–disk limit *so slowly* ? After all, Eq. (6) describes the scattering between two *typical* configurations in the wavefunctions of the incoming onia, by which I mean configurations which involve a number of dipoles N close to the average value $n(Y/2) \sim \mathrm{e}^{\omega_{\mathbb{P}} Y/2}$ and for which $P_N = \mathcal{O}(1)$. The huge difference between the estimates (7) and (6) suggests that, at high energy, S_Y is rather dominated by *rare* configurations, which involve only few gluons ($N \ll n(Y/2)$), and therefore have a very low probability to occur ($P_N(Y/2) \ll 1$), but which give a much larger contribution to S simply because systems with fewer dipoles have a smaller probability to interact, and thus a larger probability S^2 to survive without interactions. (Note that the various configurations contribute additively to the S–matrix in Eq. (5), so the sum there is dominated by those configurations which maximize the product $P_N(Y/2)P_{N'}(Y/2)S_{N\times N'}$.)

At this point, it is interesting to note that a result very similar to that in Eq. (7) is obtained from the high–energy limit of the BK equation [14], and also from approximate solutions to JIMWLK equation valid deeply at saturation [15]. Specifically, BK equation yields [14,12]

$$S_Y(r_0) \simeq \mathrm{e}^{-\frac{c}{2}\bar\alpha_s^2 (Y-Y_0)^2}\,, \tag{8}$$

where Y_0 is such that $Q_s(Y_0) \sim 1/r_0$, and $c \approx 4.88$ is the exponent giving the energy dependence of the saturation momentum [3]: $Q_s^2(Y) \simeq Q_s^2(0)\mathrm{e}^{c\bar\alpha_s Y}$. Now, in the asymmetric frame in which one dipole is bare and the other one is highly evolved, the BK equation describes the scattering between the bare dipole and the *typical* configuration in the evolved one, which is a CGC with saturation momentum $Q_s(Y)$ [15,12]. Then, the discrepancy between Eqs. (8) and (6) shows that typical configurations play very different roles in different frames, thus illustrating the strong sensitivity of the physical picture of the high–energy scattering upon the choice of a frame. In particular, the configurations retained by the BK equation in the COM frame must be some rare configurations, with only few gluons. But then it is legitimate to rise doubts about Eq. (8) too : Recall indeed that BK equation is obtained after a MFA, which should work reasonably well for *typical* configurations, but not also for the *rare* ones ! But if rare configurations play such an important role at high energies in the COM frame, there is no reason why they should be less important in the asymmetric frame. In other terms, one cannot trust any result like (6) or (8), which is obtained by including the typical configurations *alone*.

Unfortunately, there seems to be no systematic way to identify the rare configurations which dominate the S–matrix at high energies. As a rough criterion, the relevant configurations must involve the *maximal* number of gluons which can still give a contribution to S of order one: indeed, further increasing the number of gluons would rapidly decrease S, whereas reducing the number of gluons even stronger would suppress the probability of the configuration without significantly enhancing its contribution to S. With this criterion in mind, we have been able to 'guess' some optimal configurations [12], and then check that they do indeed a better job than the configurations retained by the BK equation, in the sense of giving a larger contribution to S. Specifically, our best configurations yield

$$S_Y(r_0) \,\simeq\, \mathrm{e}^{-\frac{c}{4}\bar{\alpha}_s^2(Y-Y_0)^2}\,, \tag{9}$$

where as compared to Eq. (8) the exponent is now reduced by a factor of two. Note that these optimal configurations are rare in *any* frame, and look differently in different frames [12]. In particular, in the asymmetric frame where typical configurations would lead to the result (8), the optimal, rare, configurations are those in which the fast onium has evolved into a CGC having a lower than normal saturation momentum: $Q_s((Y+Y_0)/2)$ instead of $Q_s(Y)$.

To conclude, let me notice that the exponent in Eq. (9) is still larger than the one reported in the numerical calculation in Ref. [5] (cf. Eq. (7)): $c/4 \approx 1.22$ rather than $\kappa \approx 0.72$. So, it would be worth redoing the numerical analysis in order to understand the origin of this discrepancy.

References

1. L.N. Lipatov, *Sov. J. Nucl. Phys.* **23** (1976) 338; E.A. Kuraev, L.N. Lipatov, V.S. Fadin, *Zh. Eksp. Teor. Fiz* **72**, 3 (1977) (*Sov. Phys. JETP* **45** (1977) 199); Ya.Ya. Balitsky, L.N. Lipatov, *Sov. J. Nucl. Phys.* **28** (1978) 822.
2. A. H. Mueller, *Nucl. Phys.* **B415** (1994) 373; A. H. Mueller and B. Patel, *Nucl. Phys.* **B425** (1994) 471; A. H. Mueller, *Nucl. Phys.* **B437** (1995) 107.
3. L. V. Gribov, E. M. Levin, and M. G. Ryskin, *Phys. Rept.* **100** (1983) 1.
4. L. McLerran and R. Venugopalan, *Phys. Rev.* **D49** (1994) 2233; *ibid.* 3352.
5. G.P. Salam, *Nucl. Phys.* **B449** (1995) 589; *Nucl. Phys.* **B461** (1996) 512; A. H. Mueller and G.P. Salam, *Nucl. Phys.* **B475** (1996) 293.
6. Yu. V. Kovchegov, *Phys. Rev.* **D60** (1999), 034008; *ibid.* **D61** (2000) 074018.
7. E. Iancu, A. Leonidov and L. McLerran, *Nucl. Phys.* **A692** (2001), 583; *Phys. Lett.* **B510** (2001) 133; E. Ferreiro, E. Iancu, A. Leonidov and L. McLerran, *Nucl. Phys.* **A703** (2002) 489.
8. J. Jalilian-Marian, A. Kovner, A. Leonidov and H. Weigert, *Nucl. Phys.* **B504** (1997) 415; *Phys. Rev.* **D59** (1999) 014014.
9. H. Weigert, *Nucl. Phys.* **A703** (2002) 823.
10. I. Balitsky, *Nucl. Phys.* **B463** (1996) 99; *Phys. Rev. Lett.* **81** (1998) 2024.
11. E. Iancu and A.H. Mueller, *Nucl. Phys.* **A730** (2004) 460.
12. E. Iancu and A.H. Mueller, *Nucl. Phys.* **A730** (2004) 494.
13. M. Kozlov and E. Levin, *Nucl. Phys.* **A739** (2004) 291.
14. E. Levin and K. Tuchin, *Nucl. Phys.* **B573** (2000) 833.
15. E. Iancu and L. McLerran, *Phys. Lett.* **B510** (2001) 145.