ON THE CONVERGENCE OF A SERIES MAPPED BY A FUNCTION

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Abstract. We provide a characterization of two families of real functions, namely, of those functions f such that the series $\sum f(x_n)$ diverges whenever the series $\sum x_n$ diverges, or, respectively, whenever the series $\sum x_n$ non-absolutely converges. This solves two open problems of J. Borsík. We also reformulate known results on families of functions preserving or changing the type of convergence of series, and add some results about divergent series of terms converging to zero.

1. Introduction

In his paper [1], J. Borsík studied functions $f : \mathbb{R} \to \mathbb{R}$ which map every series of some convergence type to a series of some other convergence type. More precisely, let A and B be some families of sequences of real numbers. The following families were considered in [1]:

$$
\begin{aligned}
C &= \big\{\{x_n\}_{n\in\mathbb{N}} : \textstyle\sum x_n \text{ converges}\big\},\\
AC &= \big\{\{x_n\}_{n\in\mathbb{N}} : \textstyle\sum x_n \text{ absolutely converges}\big\},\\
RC &= \big\{\{x_n\}_{n\in\mathbb{N}} : \textstyle\sum x_n \text{ relatively converges}\big\},\\
D &= \big\{\{x_n\}_{n\in\mathbb{N}} : \textstyle\sum x_n \text{ diverges}\big\},\\
C^+ &= \big\{\{x_n\}_{n\in\mathbb{N}} : x_n \geq 0 \text{ for all } n \text{ and } \textstyle\sum x_n \text{ converges}\big\},\\
D^+ &= \big\{\{x_n\}_{n\in\mathbb{N}} : x_n \geq 0 \text{ for all } n \text{ and } \textstyle\sum x_n \text{ diverges}\big\}.
\end{aligned}
$$

Let us note that the series $\sum x_n$ is relatively convergent if the series $\sum x_n$ converges but $\sum |x_n|$ diverges. Relative convergence is also called 'non-absolute convergence' or 'conditional convergence'.

Denote by $F(A, B)$ the family of all functions $f : \mathbb{R} \to \mathbb{R}$ such that for every sequence $\{x_n\}_{n\in\mathbb{N}}$ belonging to A, the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ belongs to B. For all but two combinations of $A, B \in \{C, AC, RC, D, C^+, D^+\}$, a characterization of functions belonging to $F(A, B)$ was found in [1]. The two remained cases

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are $F(D,D)$ and $F(RC,D)$. In this paper, we provide a characterization of functions belonging to these two families. We also consider one more family of sequences of real numbers,

$$D_0 = \left\{\{x_n\}_{n\in\mathbb{N}} : \lim_{n\to\infty} x_n = 0 \text{ and } \sum x_n \text{ diverges}\right\}.$$

For every $A, B \in \{C, AC, RC, D, D_0\}$ we will find a simple family $\mathcal{F}$ of closed subsets of $\mathbb{R}^2$ such that a function $f : \mathbb{R} \to \mathbb{R}$ belongs to $F(A,B)$ if and only if its graph is a subset of some $E \in \mathcal{F}$. Finally, we show how to characterize families $F(A^+, B)$ and $F(A, B^+)$, where A^+ denotes the family of all non-negative sequences belonging to A.

2. Known results

In this section, we review known results on the families $F(A,B)$, for $A, B \in \{C, RC, AC, D\}$. We also prove some statements related to the family D_0.

The characterization of $F(C,C)$ is a well known theorem of R. Rado [3].

Theorem 2.1. *$F(C,C) = \big\{f : \mathbb{R} \to \mathbb{R} : \exists a \in \mathbb{R}\ \exists \delta > 0\ \forall x \in (-\delta, \delta)\ f(x) = ax\big\}$.*

We formulate this result, as well as related results proved by J. Borsík [1], in another way, using families of simple closed subsets of the plane. For $a, b \in \mathbb{R}$, $b > 0$, denote

$$N(a,b) = \left\{(x,y) \in \mathbb{R}^2 : y = ax \ \vee\ |x| \geq b\right\}.$$

Let us note that we identify a function $f : \mathbb{R} \to \mathbb{R}$ and its graph.

Parts (1)–(3) of the following theorem are reformulations of Theorems 10, 11, and 3 of [1].

Theorem 2.2.

(1) $F(C,C) = F(RC,C) = \{f : \exists a \in \mathbb{R}\ \exists b > 0\ f \subseteq N(a,b)\}$,
(2) $F(RC,RC) = \{f : \exists a \neq 0\ \exists b > 0\ f \subseteq N(a,b)\}$,
(3) $F(C,AC) = F(RC,AC) = \{f : \exists b > 0\ f \subseteq N(0,b)\}$,
(4) $F(D_0,C) = F(D_0,AC) = \{f : \exists b > 0\ f \subseteq N(0,b)\}$.

Proof. To show (4), notice that $\{f : \exists b > 0\ f \subseteq N(0,b)\} \subseteq F(D_0,AC) \subseteq F(D_0,C)$. If $f(0) \neq 0$ then f maps every sequence containing infinitely many 0's to a divergent series, hence $f(0) = 0$ holds for every $f \in F(D_0,C)$. If there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \to 0$ and for every n, $x_n \neq 0$ and $f(x_n) \neq 0$, then there is a sequence $\{k_n\}_{n\in\mathbb{N}}$ of natural numbers such that for all n, $|x_n| \geq 1/k_n$ and $|f(x_n)| \geq 1/k_n$. If a sequence $\{y_j\}_{j\in\mathbb{N}}$ consists of k_1-many x_1's followed by k_2-many x_2's, and so on, then clearly $\{y_j\}_{j\in\mathbb{N}}$ belongs to D_0 but $\{f(y_j)\}_{j\in\mathbb{N}}$ is not in C, hence $f \notin F(D_0,C)$. It follows that $F(D_0,C) \subseteq \{f : \exists b > 0\ f \subseteq N(0,b)\}$. $\square$

For $a \geq 0$, $b > 0$, let

$$X(a,b) = \left\{(x,y) \in \mathbb{R}^2 : |y| \leq a\,|x| \ \vee \ |x| \geq b\right\},$$
$$O(b) = \left\{(x,y) \in \mathbb{R}^2 : |x| \geq b \ \vee \ |y| \geq b\right\}.$$

Let θ denote the zero function, i.e., $\theta(x) = 0$ for every $x \in \mathbb{R}$.

Parts (1)–(4) of the following theorem are reformulations of Theorems 2, 7, 5, and 4 of [1].

Theorem 2.3.

(1) $F(AC, C) = F(AC, AC) = \{f : \exists a \geq 0 \ \exists b > 0 \ f \subseteq X(a,b)\}$,
(2) $F(C, D) = F(AC, D) = \{f : \exists b > 0 \ f \subseteq O(b)\}$,
(3) $F(D, C) = F(D, AC) = \{\theta\}$,
(4) $F(C, RC) = F(AC, RC) = F(D, RC) = \emptyset$,
(5) $F(D_0, RC) = F(C, D_0) = F(AC, D_0) = F(D, D_0) = \emptyset$.

Proof. To show that $F(D_0, RC) = \emptyset$, assume that $f \in F(D_0, RC)$ and $x_n = 1/n$, for $n \in \mathbb{N}$. Put $A = \{n \in \mathbb{N} : f(x_n) \geq 0\}$, $B = \mathbb{N} \setminus A$. At least one of the series $\sum_{n\in A} x_n$, $\sum_{n\in B} x_n$ must be divergent. Since both series $\sum_{n\in A} f(x_n)$ and $\sum_{n\in B} f(x_n)$ are divergent, we have that $f \notin F(D_0, RC)$.

To see that $F(A, D_0) = \emptyset$ for $A \in \{C, AC, D\}$ it suffices to take any $f : \mathbb{R} \to \mathbb{R}$ and a constant sequence $\{x_n\}_{n\in\mathbb{N}}$ in A. Then $\{f(x_n)\}_{n\in\mathbb{N}}$ is again a constant sequence and hence it does not belong to D_0. □

3. Preserving the divergence of series

The main result of this section is the characterization of the family $F(D, D)$. By a slight modification we also obtain characterizations of $F(D_0, D)$ and $F(D_0, D_0)$.

For $a \in \mathbb{R}$ and $b > 0$, denote

$$Y(a,b) = \left\{(x,y) \in \mathbb{R}^2 : x = 0 \vee |y| \geq b \vee |x| \leq ay\right\}$$

and

$$Z(a,b) = \left\{(x,y) \in \mathbb{R}^2 : x = 0 \vee |y| \geq b \vee x = ay\right\}.$$

The family $F(D, D)$ can be then characterized as follows.

Theorem 3.1. $F(D, D) = \{f : \exists a \in \mathbb{R} \ \exists b > 0 \ (f \subseteq Y(a,b) \ \vee \ f \subseteq Z(a,b))\}$.

We divide the proof of Theorem 3.1 into several steps.

Lemma 3.2. *Let $f \subseteq Y(a,b)$ for some $a \in \mathbb{R}$ and $b > 0$. Then $f \in F(D, D)$.*

Proof. Assume that $f \subseteq Y(a,b)$, $b > 0$, and a series $\sum_{n\in\mathbb{N}} f(x_n)$ is convergent. We prove that $\sum_{n\in\mathbb{N}} x_n$ must converge, too.

Without loss of generality we may assume that $f(0) = 0$. Otherwise, $x_n \neq 0$ for all n except finitely many, and the value $f(0)$ has no impact on the convergence of $\sum_{n\in\mathbb{N}} f(x_n)$. There exists n_0 such that $|f(x_n)| < b$ for $n \geq n_0$. Hence, for all $n \geq n_0$, $x_n = 0$ or $|x_n| \leq af(x_n)$. It follows that $f(x_n)f(x_k) \geq 0$ for all $n, k \geq n_0$, and the series $\sum_{n\in\mathbb{N}} f(x_n)$ converges absolutely. Since $|x_n| \leq |a|\,|f(x_n)|$, also the series $\sum_{n\in\mathbb{N}} x_n$ converges absolutely. $\square$

Lemma 3.3. *Let $f \subseteq Z(a,b)$ for some $a \in \mathbb{R}$ and $b > 0$. Then $f \in F(D,D)$.*

Proof. Again, assume that $f \subseteq Z(a,b)$, $b > 0$, and a series $\sum_{n\in\mathbb{N}} f(x_n)$ converges. Similarly as in Lemma 3.2, we may assume that $f(0) = 0$. There exists n_0 such that for $n \geq n_0$ we have $|f(x_n)| < b$, hence $x_n = 0$ or $x_n = af(x_n)$. It follows that $x_n = af(x_n)$ holds for all $n \geq n_0$, and thus the series $\sum_{n\in\mathbb{N}} x_n$ converges. $\square$

We will need two auxiliary results.

Lemma 3.4. *Let $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ be sequences of positive reals such that*

$$\lim_{n\to\infty} a_n = 0 \text{ and } \lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

Then there exists a sequence $\{m_n\}_{n\in\mathbb{N}}$ of non-negative integers such that

$$\sum_{n\in\mathbb{N}} m_n a_n < \infty \text{ and } \sum_{n\in\mathbb{N}} m_n b_n = \infty.$$

Proof. There exists an increasing sequence of natural numbers $\{n_k\}_{k\in\mathbb{N}}$ such that the series $\sum_{k\in\mathbb{N}} a_{n_k}$ and $\sum_{k\in\mathbb{N}} a_{n_k}/\,b_{n_k}$ converge. Put $c_k = 1/\,b_{n_k}$. Then the series $\sum_{k\in\mathbb{N}} c_k a_{n_k}$ converges and the series $\sum_{k\in\mathbb{N}} c_k b_{n_k}$ diverges. Let m_n be the least integer $\geq c_k$ if $n = n_k$, and $m_n = 0$ if $n \notin \{n_k : k \in \mathbb{N}\}$. Then $\sum_{n\in\mathbb{N}} m_n a_n = \sum_{k\in\mathbb{N}} m_{n_k} a_{n_k} \geq \sum_{k\in\mathbb{N}} (c_k + 1) a_{n_k} < \infty$, and $\sum_{n\in\mathbb{N}} m_n b_n \geq \sum_{k\in\mathbb{N}} c_k b_{n_k} = \infty$. $\square$

Lemma 3.5. *Let $a, b, c, d \in \mathbb{R}$ be such that $b, d > 0$ and $a/b \neq c/d$. Then there exist $m, n \in \mathbb{N}$ such that $|ma - nc| > 1$ and $|mb - nd| < \max\{b, d\}$.*

Proof. Let us denote

$$v = \left|\frac{a}{b} - \frac{c}{d}\right|, \ u = \frac{|c|}{d} \cdot \max\{b, d\} + 1,$$

and put

$$m = \min\left\{k \in \mathbb{N} : kb > \frac{u}{v}\right\}, \ n = \min\left\{k \in \mathbb{N} : kd > \frac{u}{v}\right\}.$$

Then $mb > u/v$, $(m-1)b \leq u/v$, $nd > u/v$, $(n-1)d \leq u/v$, hence $mb - nd < (u/v + b) - u/v = b$ and $mb - nd > u/v - (u/v + d) = -d$, thus $|mb - nd| < \max\{b, d\}$.

We also have

$$ma - nc = mb \cdot \frac{a}{b} - nd \cdot \frac{c}{d} = mb \cdot \left(\frac{a}{b} - \frac{c}{d}\right) + (mb - nd) \cdot \frac{c}{d}$$

and since

$$\left|mb \cdot \left(\frac{a}{b} - \frac{c}{d}\right)\right| = mbv > u \text{ and } \left|(mb - nd) \cdot \frac{c}{d}\right| < \max\{b, d\} \cdot \frac{|c|}{d} = u - 1,$$

we obtain that $|ma - nc| > 1$. □

Lemma 3.6. *If $f \in F(D, D)$ then there exists $\varepsilon > 0$ such that for all $x, y \in \mathbb{R}$, $|f(x)| < \varepsilon \wedge |f(y)| < \varepsilon \wedge xy > 0 \Rightarrow f(x)f(y) > 0$.*

Proof. It is clear that if $f(x) = 0$ for some $x \neq 0$ then $f \notin F(D, D)$. Assume that $f(x) \neq 0$ for all $x \neq 0$ and the conclusion of the lemma does not hold, i.e., for every $\varepsilon > 0$ there exist x, y such that $|f(x)| < \varepsilon$, $|f(y)| < \varepsilon$, $xy > 0$ and $f(x)f(y) < 0$. Then there are sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ such that all x_n's and y_n's have the same sign, and for every n, $|f(x_n)| < 2^{-n}$, $|f(y_n)| < 2^{-n}$, and $f(x_n)f(y_n) < 0$.

For every n, denote $a_n = |x_n|$, $b_n = |f(x_n)|$, $c_n = -|y_n|$, $d_n = |f(y_n)|$. Since $a_n/b_n \neq c_n/d_n$, by Lemma 3.5 there exist $u_n, v_n \in \mathbb{N}$ such that $|u_n a_n - v_n c_n| > 1$ and $|u_n b_n - v_n d_n| < \max\{b_n, d_n\}$. We obtain $|u_n x_n + v_n y_n| = |u_n a_n - v_n c_n| > 1$ and $|u_n f(x_n) + v_n f(y_n)| = |u_n b_n - v_n d_n| < \max\{|f(x_n)|, |f(y_n)|\}$.

Let us define a sequence $\{z_j\}_{j\in\mathbb{N}}$ as follows. For every n by induction define a finite sequence $\{w_i^n\}_{i=1}^{u_n+v_n}$ containing u_n-times value x_n and v_n-times value y_n and such that for each $k \in \{1, \dots, u_n + v_n\}$,

$$\left|\sum_{i=1}^{k} f(w_i^n)\right| \leq \max\{|f(x_n)|, |f(y_n)|\} < 2^{-n}.$$

Let $\{z_j\}_{j\in\mathbb{N}}$ be the concatenation of sequences $\{w_i^1\}_{i=1}^{u_1+v_1}$, $\{w_i^2\}_{i=1}^{u_2+v_2}$, etc. It is easy to check that the series $\sum_{j\in\mathbb{N}} z_j$ diverges and $\sum_{j\in\mathbb{N}} f(z_j)$ converges, hence $f \notin F(D, D)$. □

Lemma 3.7. *If $f \in F(D, D)$ then there exist $a \in \mathbb{R}$ and $b > 0$ such that $f \subseteq Y(a, b)$ or $f \subseteq Z(a, b)$.*

Proof. Let $f \in F(D, D)$. Let us first assume that a statement stronger than that of Lemma 3.6 holds true, namely that there exists $\varepsilon > 0$ such that

$$\forall x, y \neq 0 \ \ |f(x)| < \varepsilon \ \wedge \ |f(y)| < \varepsilon \ \Rightarrow \ f(x)f(y) > 0. \tag{1}$$

We show that then $f \subseteq Y(a, b)$, for some $a \in \mathbb{R}$ and $b > 0$.

Assume the opposite. Then for every a and $b > 0$ there exists x_{ab} such that $(x_{ab}, f(x_{ab})) \notin Y(a, b)$, i.e., $x_{ab} \neq 0$, $|f(x_{ab})| < b$, and $|x_{ab}| > af(x_{ab})$. Let $s \in \{-1, 1\}$ be the common sign of all values $f(x)$ with $x \neq 0$ and $|f(x)| < \varepsilon$, i.e., $sf(x) > 0$ whenever $x \neq 0$ and $|f(x)| < \varepsilon$. For $b < \varepsilon$ and $sa > 0$ we have $sf(x_{ab}) > 0$, hence also $af(x_{ab}) > 0$ and $|x_{ab}| > |a| |f(x_{ab})|$. It follows that there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of nonzero reals such that $f(x_n) \to 0$ and $f(x_n)/x_n \to 0$. Moreover, there is a subsequence $\{y_n\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that all y_n's have the same sign. Using Lemma 3.4 we can find a sequence $\{m_n\}_{n\in\mathbb{N}}$ of non-negative integers such that the series $\sum_{n\in\mathbb{N}} m_n |f(y_n)|$ converges and $\sum_{n\in\mathbb{N}} m_n |y_n|$ diverges. Let $\{z_j\}_{j\in\mathbb{N}}$ be a sequence containing m_1-times value y_1, then m_2-times value y_2, etc. Since all y_n's have the same sign, we obtain that the series $\sum_{j\in\mathbb{N}} z_j$ is divergent while the series $\sum_{j\in\mathbb{N}} f(z_j)$ is absolutely convergent. This contradicts the assumption $f \in F(D, D)$.

Now, assume that (1) fails for every $\varepsilon > 0$. We show that then $f \subseteq Z(a, b)$, for some $a \in \mathbb{R}$ and $b > 0$. If not, then for every a and $b > 0$ there exists x such that $(x, f(x)) \notin Z(a, b)$, i.e., $x \neq 0$, $|f(x)| < b$, and $x \neq af(x)$. We claim that for every $b > 0$ there exist x and y such that $xy < 0$, $|f(x)| < b$, $|f(y)| < b$, $f(x)f(y) < 0$, and $f(x)/x \neq f(y)/y$.

This can be proved as follows. Let $\varepsilon > 0$ be as in Lemma 3.6. Let $s \in \{-1, 1\}$ be the common sign of all values $f(x)$ with $x > 0$ and $|f(x)| < \varepsilon$, i.e., $sf(x) > 0$ whenever $x > 0$ and $|f(x)| < \varepsilon$. Then $sf(x) < 0$ whenever $x < 0$ and $|f(x)| < \varepsilon$, and there is $s' \in \{-1, 1\}$ such that $s'xf(x) > 0$ whenever $x \neq 0$ and $|f(x)| < \varepsilon$. Take any $x \neq 0$ such that $|f(x)| < \min\{\varepsilon, b\}$. Then $f(x) \neq 0$ and there exists $y \neq 0$ such that $|f(y)| < \min\{\varepsilon, b\}$ and $y \neq (x/f(x))f(y)$. If $xy < 0$ then also $f(x)f(y) < 0$ and we are done. Otherwise, since (1) fails, there is $z \neq 0$ such that $|f(z)| < \min\{\varepsilon, b\}$ and $f(x)f(z) < 0$. Then also $xz < 0$, $yz < 0$, and $f(x)/x \neq f(z)/z$ or $f(y)/y \neq f(z)/z$.

One can now easily find sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ such that for every n, $x_n > 0$, $y_n < 0$, $|f(x_n)| < 2^{-n}$, $|f(y_n)| < 2^{-n}$, $f(x_n)f(y_n) < 0$, and $f(x_n)/x_n \neq f(y_n)/y_n$. Denote $a_n = x_n$, $b_n = |f(x_n)|$, $c_n = -y_n$, and $d_n = |f(y_n)|$. By Lemma 3.5 there exist natural numbers u_n, v_n such that $|u_n a_n - v_n c_n| > 1$ and $|u_n b_n - v_n d_n| < \max\{b_n, d_n\}$. It follows that $|u_n x_n + v_n y_n| = |u_n a_n - v_n c_n| > 1$ and $|u_n f(x_n) + v_n f(y_n)| = |u_n b_n - v_n d_n| < \max\{|f(x_n)|, |f(y_n)|\}$. Similarly as in Lemma 3.6 we can find a sequence $\{z_j\}_{j\in\mathbb{N}}$ such that the series $\sum_{j\in\mathbb{N}} z_j$ diverges and $\sum_{j\in\mathbb{N}} f(z_j)$ converges, contradicting the assumption $f \in F(D, D)$. □

Theorem 3.1 now follows directly from Lemmas 3.2, 3.3 and 3.7.

We are now going to characterize families $F(D_0, D)$ and $F(D_0, D_0)$. For any $f, g : \mathbb{R} \to \mathbb{R}$, let $f \sim g$ if and only if $\exists \varepsilon > 0\ \forall x \in \mathbb{R}\ |x| < \varepsilon \Rightarrow f(x) = g(x)$. Clearly, $\sim$ is an equivalence relation.

Theorem 3.8. *Let $f : \mathbb{R} \to \mathbb{R}$. The following conditions are equivalent.*

(1) *$f \in F(D_0, D)$,*
(2) *there exists $g \in F(D, D)$ such that $f \sim g$.*

Proof. For a function $f : \mathbb{R} \to \mathbb{R}$ and $\varepsilon > 0$, define

$$f_\varepsilon(x) = \begin{cases} f(x) & \text{if } |x| < \varepsilon, \\ \varepsilon & \text{otherwise.} \end{cases}$$

Clearly, $f_\varepsilon \sim f$. Let us assume that (2) does not hold, hence $\forall \varepsilon > 0$ $f_\varepsilon \notin F(D,D)$. Fix a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ of positive reals converging to 0. For each k, find a sequence $\{x_i^k\}_{i\in\mathbb{N}}$ such that the series $\sum_{i\in\mathbb{N}} x_i^k$ diverges but the series $\sum_{i\in\mathbb{N}} f_{\varepsilon_k}(x_i^k)$ converges. There exists p_k such such that for all $i \geq p_k$, $\left|f_{\varepsilon_k}(x_i^k)\right| < \varepsilon_k$, hence also $\left|x_i^k\right| < \varepsilon_k$ and $f(x_i^k) = f_{\varepsilon_k}(x_i^k)$. Since $\sum x_i^k$ diverges, there exists $\eta_k > 0$ such that

$$\forall m\ \exists n \geq m\ \left|\sum_{i=m}^{n} x_i^k\right| \geq \eta_k.$$

Let M_k be such that $M_k\eta_k \geq 1$, let $m_k \geq p_k$ be such that

$$\forall n \geq m_k\ \left|\sum_{i=m_k}^{n} f_{\varepsilon_k}(x_i^k)\right| \leq \frac{1}{M_k 2^k},$$

and let $n_k \geq m_k$ be such that

$$\left|\sum_{i=m_k}^{n_k} x_i^k\right| \geq \eta_k.$$

Let the sequence $\{z_j\}_{j\in\mathbb{N}}$ be created by joining M_1 copies of sequence $\{x_i^1\}_{i=m_1}^{n_1}$, then M_2 copies of sequence $\{x_i^2\}_{i=n_2}^{m_2}$, and so on. Then $\{z_j\}_{j\in\mathbb{N}}$ converges to 0, the series $\sum_{j\in\mathbb{N}} z_j$ diverges, and the series $\sum_{j\in\mathbb{N}} f(z_j)$ converges, hence $f \notin F(D_0,D)$. We have thus proved (1) $\Rightarrow$ (2).

To prove the other direction, assume that $f \sim g$ and $g \in F(D,D)$. Let $\{x_i\}_{i\in\mathbb{N}} \in D_0$ be arbitrary, and let $\varepsilon > 0$ be such that $f(x) = g(x)$ for $|x| < \varepsilon$. There exists n such that for all $i \geq n$, $|x_i| < \varepsilon$, hence $f(x_i) = g(x_i)$. The series $\sum_{i\in\mathbb{N}} g(x_i)$ diverges, hence also $\sum_{i\in\mathbb{N}} f(x_i)$ diverges. Thus, $f \in F(D_0,D)$. □

To characterize family $F(D_0,D)$, we may use an easy modification of Theorem 3.1. For $a \in \mathbb{R}$ and $b > 0$, denote

$$Y'(a,b) = \left\{(x,y) \in \mathbb{R}^2 : x = 0 \ \vee\ |x| \geq b \ \vee\ |y| \geq b \ \vee\ |x| \leq ay\right\}$$

and

$$Z'(a,b) = \left\{(x,y) \in \mathbb{R}^2 : x = 0 \ \vee\ |x| \geq b \ \vee\ |y| \geq b \ \vee\ x = ay\right\}.$$

Theorem 3.9. $F(D_0,D) = \{f : \exists a \in \mathbb{R}\ \exists b > 0\ (f \subseteq Y'(a,b) \ \vee\ f \subseteq Z'(a,b))\}$.

Proof. It follows directly from Theorems 3.1 and 3.8. □

Theorem 3.10. *Let $f : \mathbb{R} \to \mathbb{R}$. The following conditions are equivalent.*

(1) $f \in F(D_0,D_0)$,
(2) $f \in F(D_0,D) \wedge \lim_{x\to 0} f(x) = f(0) = 0$.

Proof. To show that (1) implies $\lim_{x\to 0} f(x) = 0$, assume that $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of non-zero reals converging to 0, $c > 0$, and for all n, $|f(x_n)| \geq c$. For each n, let $m_n \in \mathbb{N}$ be such that $|m_n x_n| \geq 1$. Then the sequence containing m_1-times value x_1, then m_2-times value x_2, and so on, witnesses that $f \notin F(D_0, D_0)$. □

For characterization of family $F(D_0, D_0)$ in a form similar to Theorem 3.9 we use a fact that for every function f such that $\lim_{x\to 0} f(x) = f(0) = 0$ there exists a continuous function h satisfying the same condition and such that $h(x) \geq f(x)$ holds on some neighborhood of 0. Indeed, let $\{x_n\}_{n\in\mathbb{N}}$ be a decreasing sequence converging to 0 and such that f is bounded on $(-x_0, x_0)$. One can take h such that $h(0) = 0$, $h(x_{n+1}) = h(-x_{n+1}) = \sup\{|f(x)| : x \in [-x_n, x_n]\}$ for every n, and between these points it is defined linearly.

Let $\mathcal{U}$ be the family of all continuous functions h such that $\lim_{x\to 0} h(x) = h(0) = 0$. For $h \in \mathcal{U}$, denote

$$U(h) = \left\{(x, y) \in \mathbb{R}^2 : |y| \leq h(x)\right\}.$$

Clearly, each $U(h)$ is closed.

From Theorems 3.9 and 3.10 be obtain a characterization of $F(D_0, D_0)$ by a family of closed subsets of the plane.

Corollary 3.11. *$F(D_0, D_0) = \{f : \exists a \in \mathbb{R}\ \exists b > 0\ \exists h \in \mathcal{U}\ (f \subseteq Y'(a, b) \cap U(h)\ \vee\ f \subseteq Z'(a, b) \cap U(h))\}$.*

4. Mapping relatively convergent series to divergent series

In this section we characterize families $F(RC, D)$ and $F(RC, D_0)$. One natural example of a function belonging to $F(RC, D)$ is $f(x) = |x|$. Let us mention another, more interesting example (see [1], Example 1). Let $a, b \in \mathbb{R}$, and let $g(x) = ax$ if $x \geq 0$, $g(x) = bx$ otherwise. Then $g \in F(RC, D)$ if and only if $a \neq b$.

To formulate our result, we will need some notation. For $f : \mathbb{R} \to \mathbb{R}$ and $\varepsilon > 0$, denote

$$R_f^-(\varepsilon) = \left\{\frac{f(x)}{x} : x \in (-\varepsilon, 0)\ \wedge\ |f(x)| < \varepsilon\right\}$$

and

$$R_f^+(\varepsilon) = \left\{\frac{f(x)}{x} : x \in (0, \varepsilon)\ \wedge\ |f(x)| < \varepsilon\right\}.$$

For $a, b, c, d \in \mathbb{R}$, where $b, c > 0$, $d \in \{-1, 1\}$, denote

$$K(a, b, c, d) = \left\{(x, y) \in \mathbb{R}^2 : x = 0\ \vee\ |x| \geq c\ \vee\ |y| \geq c\ \vee\ dy \geq ax + b\,|x|\,\right\},$$

$$L(c, d) = \left\{(x, y) \in \mathbb{R}^2 : |y| \geq c\ \vee\ dx \leq 0\ \vee\ dx \geq c\right\}.$$

The following theorem characterizes family $F(RC, D)$.

Theorem 4.1. *Let $f : \mathbb{R} \to \mathbb{R}$. The following conditions are equivalent.*

(1) $f \in F(RC, D)$,
(2) *there exists* $\varepsilon > 0$ *such that* $R_f^-(\varepsilon) = \emptyset$ *or* $R_f^+(\varepsilon) = \emptyset$ *or* $\inf R_f^-(\varepsilon) > \sup R_f^+(\varepsilon)$ *or* $\inf R_f^+(\varepsilon) > \sup R_f^-(\varepsilon)$,
(3) *there exist* $a \in \mathbb{R}$, $b, c > 0$, $d \in \{-1, 1\}$ *such that* $f \subseteq K(a, b, c, d)$ *or* $f \subseteq L(c, d)$.

The proof of Theorem 4.1 is based on the following lemma. Let $\mathrm{d}(A, B)$ denote the distance of two sets $A, B \subseteq \mathbb{R}$, i.e., $\mathrm{d}(A, B) = \inf\{|x - y| : x \in A \wedge y \in B\}$.

Lemma 4.2. *Let $f : \mathbb{R} \to \mathbb{R}$, $\varepsilon \in (0, 1)$ be such that $R_f^-(\varepsilon) \neq \emptyset$ and $R_f^+(\varepsilon) \neq \emptyset$.*

(a) *If* $\mathrm{d}\Big(R_f^-(\varepsilon), R_f^+(\varepsilon)\Big) = 0$ *then there exists a sequence* $\{x_i\}_{i=1}^k$ *such that* $\sum_{i=1}^k |x_i| \geq 1$ *and for* $j \in \{1, \ldots, k\}$, $\left|\sum_{i=1}^j x_i\right| < \varepsilon$ *and* $\left|\sum_{i=1}^j f(x_i)\right| < 2\varepsilon$.
(b) *If* $\mathrm{d}\Big(R_f^-(\varepsilon), R_f^+(\varepsilon)\Big) > 0$, $\inf R_f^-(\varepsilon) < \sup R_f^+(\varepsilon)$, $\inf R_f^+(\varepsilon) < \sup R_f^-(\varepsilon)$ *then there exists a sequence* $\{x_i\}_{i=1}^k$ *such that* $\sum_{i=1}^k |x_i| \geq 1$ *and for every* $j \in \{1, \ldots, k\}$, $\left|\sum_{i=1}^j x_i\right| < \varepsilon$ *and* $\left|\sum_{i=1}^j f(x_i)\right| < 3\varepsilon$.

Proof. (a) If $\mathrm{d}\Big(R_f^-(\varepsilon), R_f^+(\varepsilon)\Big) = 0$ then we can find $u \in R_f^-(\varepsilon)$, $v \in R_f^+(\varepsilon)$ such that $|u - v| \leq \varepsilon/2$. There exist $a \in (-\varepsilon, 0)$, $b \in (0, \varepsilon)$ such that $|f(a)| < \varepsilon$, $|f(b)| < \varepsilon$, $u = f(a)/a$, and $v = f(b)/b$. Without a loss of generality we may assume $|a| \leq |b|$, otherwise we can take function $g(x) = f(-x)$ instead of f.

Let $m, n \in \mathbb{N}$ be such that $1 \leq m\,|a| < 1 + |a|$ and $|ma + bn| \leq b/2$. Put $k = m + n$. We will define sequence $\{x_i\}_{i=1}^k$ by induction as follows. Let $j \in \mathbb{N}$, $1 \leq j \leq k$. If $|\{i < j : x_i = a\}| < m$ and $\left|\sum_{i<j} x_i + a\right| < b$, put $x_j = a$, otherwise put $x_j = b$.

We will show that $\left|\sum_{i=1}^j x_i\right| < b$ holds true for every $j \in \{1, \ldots, k\}$. Otherwise there exists j such that $\left|\sum_{i<j} x_i\right| < b$ and $\left|\sum_{i<j} x_i + x_j\right| \geq b$. From the definition of x_j it follows that $x_j = b > 0$. This implies $\sum_{i<j} x_i > 0$, hence $\left|\sum_{i<j} x_i + a\right| < b$, and thus $|\{i < j : x_i = a\}| = m$. We obtain that $x_i = b$ for every $j \leq i \leq k$, hence $ma + nb = \sum_{i=1}^k x_i = \sum_{i\leq j} x_i + (k - j)b$, and thus $|ma + nb| \geq \left|\sum_{i\leq j} x_i\right| \geq b$, that contradicts the assumption $|ma + nb| \leq b/2$.

For $j \in \{1, \ldots, k\}$ denote $m_j = |\{i \leq j : x_i = a\}|$. Then

$$\sum_{i=1}^{j} f(x_i) = \frac{f(b)}{b} \cdot \sum_{i=1}^{j} x_i + m_j a \left(\frac{f(a)}{a} - \frac{f(b)}{b}\right),$$

and since $\left|\sum_{i=1}^{j} x_i\right| \leq b$ and $|m_j a| \leq m\,|a| < 1 + |a|$, we have $\left|\sum_{i=1}^{j} f(x_i)\right| \leq |f(b)| + (1 + |a|)\,|u - v| < \varepsilon + (1+\varepsilon)\varepsilon/2 < 2\varepsilon$.

Since $\sum_{i=1}^{k} |x_i| = m\,|a| + n\,|b|$ and $m\,|a| \geq 1$, we can see that $\sum_{i=1}^{k} |x_i| \geq 1$.

(b) Since $\mathrm{d}\Big(R_f^-(\varepsilon), R_f^+(\varepsilon)\Big) > 0$, we have $\sup R_f^-(\varepsilon) \neq \sup R_f^+(\varepsilon)$. Therefore, either

(i) $\inf R_f^+(\varepsilon) < \sup R_f^-(\varepsilon) < \sup R_f^+(\varepsilon)$, or
(ii) $\inf R_f^-(\varepsilon) < \sup R_f^+(\varepsilon) < \sup R_f^-(\varepsilon)$.

There are $u < v < w$ such that in case (i), $u, w \in R_f^+(\varepsilon)$ and $v \in R_f^-(\varepsilon)$, and in case (ii), $u, w \in R_f^-(\varepsilon)$ and $v \in R_f^+(\varepsilon)$. In both cases there exist a, b, c such that $f(a)/a < f(b)/b < f(c)/c$, $ab < 0$, $bc < 0$, $|a| < \varepsilon$, $|b| < \varepsilon$, $|c| < \varepsilon$, $|f(a)| < \varepsilon$, $|f(b)| < \varepsilon$, $|f(c)| < \varepsilon$.

Let us denote $a' = f(a)$, $b' = f(b)$, $c' = f(c)$, and consider the equations

$$ax + by + cz = 0 \tag{2}$$

$$a'x + b'y + c'z = 0 \tag{3}$$

of variables x, y, and z.

It can be easily checked that triples of the form

$$\left(x,\ -\frac{a'c - ac'}{b'c - bc'}\,x,\ \frac{a'b - ab'}{b'c - bc'}\,x\right) \tag{4}$$

are solutions of both equations. From inequalities $ab < 0$, $bc < 0$, $a'/a < b'/b < c'/c$ we obtain that coefficients in (4) are well defined and positive.

There exist $m, n, l \in \mathbb{N}$ such that

$$m\,|a| \geq 1, \quad \left|n + m\frac{a'c - ac'}{b'c - bc'}\right| \leq \frac{1}{2}\,, \text{ and } \left|l - m\frac{a'b - ab'}{b'c - bc'}\right| \leq \frac{1}{2}\,.$$

The triple (m, n, l) is an 'approximate solution' of equations (2), (3). More precisely, since the triple

$$T = \left(m,\ -m\frac{a'c - ac'}{b'c - bc'},\ m\frac{a'b - ab'}{b'c - bc'}\right)$$

is a solution of the equations (2), (3), we have

$$ma + nb + lc = b\left(n + m\frac{a'c - ac'}{b'c - bc'}\right) + c\left(l - bm\frac{a'b - ab'}{b'c - bc'}\right) \text{ and}$$

$$ma' + nb' + lc' = b'\left(n + m\frac{a'c - ac'}{b'c - bc'}\right) + c'\left(l - bm\frac{a'b - ab'}{b'c - bc'}\right),$$

hence

$$|ma+nb+lc| \leq \frac{|b|}{2}+\frac{|c|}{2} < \varepsilon \text{ and}$$
$$|ma'+nb'+lc'| \leq \frac{|b'|}{2}+\frac{|c'|}{2} < \varepsilon.$$

Let $K \subseteq \mathbb{R}^3$ be the union of those unit cubes

$$\{(x,y,z) \in \mathbb{R}^3 : p \leq x \leq p+1 \wedge q \leq y \leq q+1 \wedge r \leq z \leq r+1\}$$

where $p,q,r \in \mathbb{Z}$, which have nonempty intersection with the line segment OT, where $O=(0,0,0)$. Since K is a connected set and $(m,n,l) \in K$, there exists a path going from O to point (m,n,l) through vertices of the unit cubes contained in K. Let $\{P_i\}_{i=0}^k$ be the shortest such path. Then for every $i \in \{1,\ldots,k\}$, $P_i - P_{i-1}$ is either $(1,0,0)$, $(0,1,0)$, or $(0,0,1)$. It follows that $k=m+n+l$.

Let us define sequence $\{x_i\}_{i=1}^k$ as follows. For $i \in \{1,\ldots,k\}$, let

$$x_i = \begin{cases} a & \text{if } P_i - P_{i-1} = (1,0,0), \\ b & \text{if } P_i - P_{i-1} = (0,1,0), \\ c & \text{if } P_i - P_{i-1} = (0,0,1). \end{cases}$$

For every $j \in \{1,\ldots,k\}$ we have $\sum_{i=1}^j x_i = ap+bq+cr$ and $\sum_{i=1}^j f(x_i) = a'p+b'q+c'r$, where $(p,q,r)=P_j$. Since $P_j \in K$, there exists a point $(x,y,z) \in OT$ such that $|p-x| \leq 1$, $|q-y| \leq 1$, and $|r-z| \leq 1$. Point (x,y,z) is a solution of equations (2), (3), hence

$$\left|\sum_{i=1}^j x_i\right| = |ap+bq+cr| \leq |a|+|b|+|c| < 3\varepsilon, \text{ and}$$
$$\left|\sum_{i=1}^j f(x_i)\right| = |a'p+b'q+c'r| \leq |a'|+|b'|+|c'| < 3\varepsilon.$$

Finally, we have $\sum_{i=1}^k |x_i| = m\,|a| + n\,|b| + l\,|c| \geq 1$. □

Proof of Theorem 4.1. (1) ⇒ (2). Assume that for every $\varepsilon > 0$, we have $R_f^-(\varepsilon) \neq \emptyset$, $R_f^+(\varepsilon) \neq \emptyset$, $\inf R_f^-(\varepsilon) \leq \sup R_f^+(\varepsilon)$, and $\inf R_f^+(\varepsilon) \leq \sup R_f^-(\varepsilon)$. There is either $\mathrm{d}\left(R_f^-(\varepsilon), R_f^+(\varepsilon)\right) = 0$, or $\mathrm{d}\left(R_f^-(\varepsilon), R_f^+(\varepsilon)\right) > 0$. In the latter case we have $\inf R_f^-(\varepsilon) < \sup R_f^+(\varepsilon)$ and $\inf R_f^+(\varepsilon) < \sup R_f^-(\varepsilon)$. Therefore, Lemma 4.2 can be applied for every $\varepsilon > 0$.

Fix a convergent series $\sum_{n\in\mathbb{N}} \varepsilon_n$ of positive reals. For every n, using Lemma 4.2 choose a sequence $\{x_i^n\}_{i=1}^{k_n}$ such that $\sum_{i=1}^{k_n} |x_i^n| \geq 1$ and for $j \in \{1,\ldots,k_n\}$, $\left|\sum_{i=1}^j x_i^n\right| < \varepsilon_n$ and $\left|\sum_{i=1}^j f(x_i^n)\right| < 3\varepsilon_n$.

Let us concatenate sequences $\{x_i^n\}_{i=1}^{k_n}$, $n \in \mathbb{N}$, into one sequence $\{z_j\}_{j\in\mathbb{N}}$, that is, if $j = \sum_{n=1}^{m-1} k_n + i$ where $1 \leq i \leq k_m$, then $z_j = x_i^m$. Clearly, the series

$\sum_{j\in\mathbb{N}} z_j$ is relatively convergent and the series $\sum_{j\in\mathbb{N}} f(z_j)$ is convergent, hence $F \notin F(RC, D)$.

(2) ⇒ (3). If $c > 0$ is such that $R^-_f(c) = \emptyset$ then for every $x \in (-c, 0)$ we have $|f(x)| \geq c$. This implies that $f \subseteq L(c, -1)$. Similarly, if $R^+_f(c) = \emptyset$ for some $c > 0$ then $f \subseteq L(c, 1)$.

If $c > 0$ is such that $\inf R^-_f(c) > \sup R^+_f(c)$ then there exist $a, b \in \mathbb{R}$ such that $b > 0$, $\sup R^+_f(c) < a - b$ and $a + b < \inf R^-_f(c)$. If $x \in (0, c)$ and $|f(x)| < c$ then $f(x)/x \leq \sup R^+_f(c) < a - b$, hence $f(x) < ax - bx$. Similarly, if $x \in (-c, 0)$ and $|f(x)| < c$ then $f(x)/x \geq \inf R^-_f(c) > a+b$, hence $f(x) < ax+bx$. Thus if $x \neq 0$, $|x| < c$ and $|f(x)| < c$ then $f(x) < ax-b\,|x|$. We obtain that $f \subseteq K(-a, b, c, -1)$.

Finally, if $\inf R^+_f(c) > \sup R^-_f(c)$ for some $c > 0$ then there exist $a, b \in \mathbb{R}$ such that $b > 0$, $\sup R^-_f(c) < a - b$ and $a + b < \inf R^+_f(c)$. For $x \in (0, c)$ such that $|f(x)| < c$ we obtain $f(x)/x \geq \inf R^+_f(c) > a+b$, hence $f(x) > ax+bx$. Similarly, for $x \in (-c, 0)$ such that $|f(x)| < c$ we have $f(x)/x \leq \sup R^-_f(c) < a - b$, hence $f(x) > ax - bx$. Thus if $x \neq 0$, $|x| < c$ and $|f(x)| < c$ then $f(x) > ax + b\,|x|$. Hence, $f \subseteq K(a, b, c, 1)$.

(3) ⇒ (1). Let $\sum_{n\in\mathbb{N}} x_n$ be a relatively convergent series, and let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying condition (3). We have to show that the series $\sum_{n\in\mathbb{N}} f(x_n)$ is divergent. The series $\sum_{n\in\mathbb{N}} x_n$ has infinitely many nonzero terms. Inserting zero terms into a series will not change its convergence and inserting finitely or infinitely many times a nonzero term a will not change a divergent series into a convergent one. Therefore, without a loss of generality, we can assume that $x_n \neq 0$ for all $n \in \mathbb{N}$. We may also assume that $\lim_{n\to\infty} f(x_n) = 0$.

If $f \subseteq K(a, b, c, 1)$ for some $a \in \mathbb{R}$, $b, c > 0$, then there exists $n_0 \in \mathbb{N}$ such that $|x_n| < c$ and $|f(x_n)| < c$, for every $n \geq n_0$. By the definition of $K(a, b, c, d)$, for $n \geq n_0$ we have $f(x_n) \geq ax_n + b\,|x_n|$. Then $\sum_{n\in\mathbb{N}} f(x_n) = \infty$ because $\sum_{n\in\mathbb{N}} b\,|x_n| = \infty$ and the set of partial sums $\left\{\sum_{n<m} ax_n : m \in \mathbb{N}\right\}$ is bounded.

Similarly, if $f \subseteq K(a, b, c, -1)$ for some $a \in \mathbb{R}$, $b, c > 0$, then there exists n_0 such that $f(x_n) \leq -ax_n - b\,|x_n|$ for every $n \geq n_0$. Then $\sum_{n\in\mathbb{N}} f(x_n) = -\infty$ because $\sum_{n\in\mathbb{N}} -b\,|x_n| = -\infty$ and the set of partial sums $\left\{\sum_{n<m} -ax_n : m \in \mathbb{N}\right\}$ is bounded.

If $f \subseteq L(c, d)$ for some $c > 0$, $d \in \{-1, 1\}$ and the series $\sum_{n\in\mathbb{N}} x_n$ relatively converges, then $|f(x_n)| \geq c$ holds true for infinitely many n, hence the series $\sum_{n\in\mathbb{N}} f(x_n)$ diverges. □

We finish this section with the characterization of the family $F(RC, D_0)$.

Theorem 4.3. *Let $f : \mathbb{R} \to \mathbb{R}$. The following conditions are equivalent.*

(1) $f \in F(RC, D_0)$,
(2) $f \in F(RC, D) \wedge \lim_{x\to 0} f(x) = f(0) = 0$.

Proof. We will show that $\lim_{x\to 0} f(x) = 0$ for every function $f \in F(RC, D_0)$. Assume that there exist $c > 0$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ of nonzero reals converging to 0 such that $|f(x_n)| \geq c$ for all $n \in \mathbb{N}$. For every n, let $m_n \in \mathbb{N}$ be such that $m_n |x_n| \geq 1$. Then the sequence $\{z_j\}_{j\in\mathbb{N}}$, containing m_1-times pair x_1, $-x_1$, then m_2-times pair x_2, $-x_2$, and so on, is such that series $\sum_{j\in\mathbb{N}} z_j$ is relatively convergent, but $\{f(z_j)\}_{j\in\mathbb{N}}$ does not converge to 0, hence $f \notin F(RC, D_0)$. □

Corollary 4.4. $F(RC, D_0) = \{f : \exists a \in \mathbb{R}\ \exists b > 0\ \exists c > 0\ \exists d \in \{-1, 1\}\ \exists g \in \mathcal{U}$ $(f \subseteq K(a, b, c, d) \cap U(g)\ \vee\ f \subseteq L(c, d) \cap U(g))\}$.

5. Series of non-negative terms

We complement presented results with two general statements on families of the forms $F(A^+, B)$ and $F(A, B^+)$. Let us recall that for a family of sequences A we define $A^+ = \big\{\{x_n\}_{n\in\mathbb{N}} \in A : \forall n\ x_n \geq 0\big\}$.

For any family A of sequences of real numbers, denote by E_A the set of all elements of sequences belonging to A, i.e.,

$$E_A = \big\{x : \exists\, \{x_n\}_{n\in\mathbb{N}} \in A\ \exists n \in \mathbb{N}\ x_n = x\big\}.$$

The following statement generalizes Proposition 2 of [1].

Theorem 5.1. *Let A, B be families of sequences of real numbers. For every function f, the following conditions are equivalent.*

(1) $f \in F(A, B^+)$,

(2) $f \in F(A, B)$ *and* $\forall x \in E_A\ f(x) \geq 0$.

Proof. If there is $x \in E_A$ such that $f(x) < 0$ then there exists a sequence in A containing x which is then mapped by f to a sequence not belonging to B^+. This proves that (1) implies that $\forall x \in E_A\ f(x) \geq 0$. The rest of the proof is trivial. □

Theorem 5.2. *Let A, B be families of sequences of real numbers. Assume that there exists a non-negative function $h \in F(A, A)$ such that $h(x) = x$ for all $x \geq 0$. Then for every function f, the following conditions are equivalent.*

(1) $f \in F(A^+, B)$,

(2) *there exists* $g \in F(A, B)$ *such that* $\forall x \geq 0\ f(x) = g(x)$.

Proof. Let the function h be as above. For $f \in F(A^+, B)$, put $g(x) = f(h(x))$, for all $x \in \mathbb{R}$. Clearly $f(x) = g(x)$ for all $x \geq 0$. Let $\{x_n\}_{n\in\mathbb{N}} \in A$ be arbitrary. Then $\{h(x_n)\}_{n\in\mathbb{N}} \in A^+$ and thus $\{g(x_n)\}_{n\in\mathbb{N}} \in B$. It follows that $g \in F(A, B)$ and hence (1) implies (2). The opposite direction is immediate. □

Corollary 5.3. *Let A, B be families of sequences of real numbers. Let $\mathcal{F}$ be a family of subsets of $\mathbb{R}^2$ such that $F(A, B) = \{f : \exists E \in \mathcal{F}\ f \subseteq E\}$.*

(1) $F(A, B^+) = \big\{f : \exists E \in \mathcal{F}\ f \subseteq E \setminus \big(E_A \times (-\infty, 0)\big)\big\}$.

(2) *If there is a non-negative function* $h \in F(A, A)$ *such that* $\forall x \geq 0\ h(x) = x$ *then* $F(A^+, B) = \big\{f : \exists E \in \mathcal{F}\ f \subseteq E \cup \big((-\infty, 0) \times \mathbb{R}\big)\big\}$.

Let us note that the assumption about the existence of function h is fulfilled for $A \in \{C, D, D_0\}$. For family C we can take $h(x) = 0$ whenever $x < 0$, for families D and D_0 we can use $h(x) = |x|$.

Corollary 5.3 allows us to characterize $F(A, B)$ for any combination of families $A, B \in \{C, AC, RC, D, D_0, C^+, D^+, D_0^+\}$.

For example, for $A = B = D$, from (1) and Theorem 3.1 we obtain $F(D, D^+) = \big\{f : \exists a \in \mathbb{R}\ \exists b > 0\ f \subseteq Y(a, b) \setminus \big(\mathbb{R} \times (-\infty, 0)\big) \ \vee\ f \subseteq Z(a, b) \setminus \big(\mathbb{R} \times (-\infty, 0)\big)\big\}$. Since every function f with graph below $Z(a, b) \setminus \big(\mathbb{R} \times (-\infty, 0)\big)$ is a subset of $Y(|a|, b) \setminus \big(\mathbb{R} \times (-\infty, 0)\big)$, we have

$$\begin{aligned} F(D, D^+) &= \big\{f : \exists a \in \mathbb{R}\ \exists b > 0\ f \subseteq Y(a, b) \setminus \big(\mathbb{R} \times (-\infty, 0)\big)\big\} \\ &= \big\{f : \exists a > 0\ \exists b > 0\ f \subseteq V(a, b)\big\}, \end{aligned}$$

where $V(a, b) = \big\{(x, y) \in \mathbb{R}^2 : y \geq a\,|x| \ \vee\ y \geq b\big\}$. Further, from (2) we obtain

$$\begin{aligned} F(D^+, D^+) &= \big\{f : \exists a \in \mathbb{R}\ \exists b > 0\ f \subseteq V(a, b) \cup \big((-\infty, 0) \times \mathbb{R}\big)\big\} \\ &= \big\{f : \exists a > 0\ \exists b > 0\ V'(a, b)\big\}, \end{aligned}$$

where $V'(a, b) = \big\{(x, y) \in \mathbb{R}^2 : x < 0 \ \vee\ y \geq ax \ \vee\ y \geq b\big\}$. These results correspond to characterizations of families $F(D, D^+)$ and $F(D^+, D^+)$ obtained in [1] (see Theorem 13 and a remark before Problem 1).

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