On Inclusions Between Arbault Sets

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Received 11. March 2003

We show that two Arbault sets characterized by increasing sequences of natural numbers are in inclusion if and only if one of these sequences is derived from another in a special way.

A set $X \subseteq \mathbb{R}$ is called an *Arbault set* if there exists an increasing sequence $a \in \mathbb{N}^{\mathbb{N}}$ such that for all $x \in X$,

$$\lim_{n \to \infty} \sin \pi a(n)\, x = 0.$$

J. Arbault considered this kind of sets when he studied the sets of absolute convergence of trigonometric series [1]. We denote by $\mathscr{A}$ the family of all Arbault sets.

Here are some properties of the family $\mathscr{A}$ (for more, see e.g. [2]).

Proposition 1. (1) *$\mathscr{A} \subseteq \mathscr{M} \cap \mathscr{N}$, where $\mathscr{M}$ and $\mathscr{N}$ denote the ideals of all meager and null sets, respectively;*

(2) *$\mathscr{A}$ contains all countable subsets of $\mathbb{R}$,*

(3) *$\mathscr{A}$ is invariant, i.e. if $X \in \mathscr{A}$ and $u, v \in \mathbb{R}$ then $\{ux + v : x \in X\} \in \mathscr{A}$;*

(4) *if $X \in \mathscr{A}$ and G is a subgroup of $(\mathbb{R}, +)$ generated by X then $G \in \mathscr{A}$;*

(5) *$\mathscr{A}$ is not an ideal.*

For given $a \in \mathbb{N}^{\mathbb{N}}$, we denote

$$A_a = \left\{x : \lim_{n \to \infty} \sin \pi a(n)\, x = 0\right\}.$$

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Key words and phrases. Arbault sets, A-sets, thin sets, characterizing sequences.
This work was supported by grant of Slovak Grant Agency VEGA 2/7555/20.

Our aim is to answer the following question: *when $A_a \subseteq A_b$?* This question was originally motivated by the study of "A-permitted" sets (see e.g. [4]).

Let us denote

$$S = \left\{ a \in \mathbb{N}^{\mathbb{N}} : a \text{ is increasing} \wedge a(0) = 1 \wedge \lim_{n\to\infty} \frac{a(n)}{a(n+1)} = 0 \right\}.$$

It is easy to see that the family $\{A_a : a \in S\}$ is a base of $\mathscr{A}$, i.e. for every $X \in \mathscr{A}$ there exists $a \in S$ such that $X \subseteq A_a$. Let us note that the condition $\lim_{n\to\infty} a(n)/a(n+1) = 0$ implies that the set A_a intersects any non-empty open set in a set of the size continuum [3].

We will answer our question for $a, b \in S$. Before we will do it, we introduce some notions.

Let $m \in \mathbb{Z}$ and $a \in S$. We say that $z \in \mathbb{Z}^{\mathbb{N}}$ is an *expansion of m by a* if

$$m = \sum_{n\in\mathbb{N}} z(n)\, a(n).$$

This of course implies that z has only finitely many non-zero elements. Further, we say that z is a *good expansion* if moreover for all $n \in \mathbb{N}$,

$$\left| \sum_{j<n} z(j)\, a(j) \right| \leq \frac{a(n)}{2}.$$

Lemma 2. *For all $m \in \mathbb{Z}$ and $a \in S$, there exists a good expansion of m by a.*

Proof. We show how to find a good expansion $z \in \mathbb{Z}^{\mathbb{N}}$. First, find some $k \in \mathbb{N}$ such that $|m| \leq a(k)/2$, and put $z(n) = 0$ for all $n > k$. Denote $m_{k+1} = m$. By an induction on n going from k to 0, define $z(n)$ to be the nearest integer to $m_{n+1}/a(n)$, and put $m_n = m_{n+1} - z(n)\, a(n)$. Since $a(0) = 1$, we obtain $m_0 = 0$, and thus for all $n \leq k$, $m_{n+1} = \sum_{j \leq k} z(j)\, a(j)$. Clearly $\sum_{n\in\mathbb{N}} z(n)\, a(n) = \sum_{n\leq k} z(n)\, a(n) = m_{k+1} = m$. For $n \leq k$ we have $|m_{n+1}/a(n) - z(n)| \leq 1/2$, hence

$$\left| \sum_{j<n} z(j)\, a(j) \right| = |m_{n+1} - z(n)\, a(n)| \leq \frac{a(n)}{2}.$$

Since a is increasing, for $n > k$ we obtain

$$\left| \sum_{j<n} z(j)\, a(j) \right| = |m| \leq \frac{a(k)}{2} < \frac{a(n)}{2}. \qquad \square$$

Let us note that we did not use the third condition from the definition of the set S. It will be used later, in the proof of Theorem 4.

A good expansion of m by a is not necessarily unique. In the previous proof, there may exist two nearest integers to $m_{n+1}/a(n)$ for some n. Any choice then leads to a good expansion. It can be however proved that this is the only case of non-uniqueness.

The following lemma shows that for a fixed a, all good expansions by a are bounded by a function depending only on a.

Lemma 3. *If z is a good expansion by a, then*

$$|z(n)| \leq \frac{1}{2}\left(1 + \frac{a(n+1)}{a(n)}\right)$$

for all $n \in \mathbb{N}$.

Proof. For a fixed n, we have

$$|z(n)\, a(n)| \leq \left|\sum_{j<n} z(j)\, a(j)\right| + \left|\sum_{j\leq n} z(j)\, a(j)\right| \leq \frac{a(n)}{2} + \frac{a(n+1)}{2},$$

hence

$$|z(n)| \leq a(n)^{-1}\left(\frac{a(n)}{2} + \frac{a(n+1)}{2}\right) = \frac{1}{2}\left(1 + \frac{a(n+1)}{a(n)}\right). \qquad \square$$

Now we are ready to formulate our result.

Theorem 4. *Let $a, b \in S$. For $k \in \mathbb{N}$, let $z_k \in \mathbb{Z}^{\mathbb{N}}$ be a good expansion of $b(k)$ by a. Then $A_a \subseteq A_b$ if and only if*
(1) $\forall n \in \mathbb{N}\ \forall^\infty k \in \mathbb{N}\ z_k(n) = 0$, *and*
(2) $\exists m \in \mathbb{N}\ \forall k \in \mathbb{N}\ \sum_{n\in\mathbb{N}} |z_k(n)| \leq m$.

We will now prove the easier direction of this theorem.

Proof of $(1) \wedge (2) \rightarrow A_a \subseteq A_b$. Assume that (1) and (2) hold true. By (2), there exists $m > 0$ such that for all k, $\sum_{n\in\mathbb{N}} |z_k(n)| \leq m$. If $x \in A_a$, and if $\varepsilon > 0$ is given, then there exists n_0 such that for all $n \geq n_0$, $|\sin \pi a(n)\, x| \leq \varepsilon/m$. By the condition (1), there exists k_0 such that for all $n < n_0$ and $k \geq k_0$, $z_k(n) = 0$. If $k \geq k_0$, then

$$|\sin \pi b(k)\, x| \leq \sum_{n\in\mathbb{N}} |z_k(n)|\, |\sin \pi a(n)\, x| \leq \frac{\varepsilon}{m} \sum_{n\in\mathbb{N}} |z_k(n)| \leq \varepsilon,$$

and hence $x \in A_b$. $\square$

In the proof of the other direction we will use the following notation: for $x \in \mathbb{R}$, let $\|x\|$ denote the distance from x to the nearest integer. It is clear that the sequence $\{\sin \pi a(n)\, x\}_{n\in\mathbb{N}}$ converges to 0 if and only if the sequence $\{\|a(n)\, x\|\}_{n\in\mathbb{N}}$ does. Also $\|-x\| = \|x$ and $\|x\| - \ |y\| \leq \|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in \mathbb{R}$.

The proof will go as follows. Assume that $(1) \wedge (2)$ is false. We define a sequence $\{I_n\}_{n\in\mathbb{N}}$ of closed intervals such that for all $n \in \mathbb{N}$,

(i) $I_{n+1} \subseteq I_n$,
(ii) the length of I_n is $4/(3a(n))$,
(iii) for all $x \in I_{n+1}$, $\|a(n)\, x\|$ is "small" and $\left\|\sum_{j\leq n} z_k(j)\, a(j)\, x\right\|$ is "big", for some selected k.

Then we will take $x \in \bigcap_{n\in\mathbb{N}} I_n$ and show that $x \in A_a \setminus A_b$.

Through the following lemmas, it is assumed that $a \in S$, $z \in \mathbb{Z}^{\mathbb{N}}$ is a good expansion by a, and some $n \in \mathbb{N}$ is fixed. We denote by $\lambda(I)$ the length of an interval I.

Lemma 5. *Let $a(n)/a(n+1) \le 1/4$. Then for every interval I such that $\lambda(I) = 4/(3a(n))$ there exists an interval $J \subseteq I$ such that $\lambda(J) = 4/(3a(n+1))$ and for all $x \in J$,*

$$\|a(n)\,x\| \le \frac{2a(n)}{3a(n+1)}.$$

Proof. Let I' be an interval of the length $1/a(n)$ co-centric with I. There exists $x_0 \in I'$ such that $\|a(n)\,x_0\| = 0$. Let J be an interval of the length $4/(3a(n+1))$ with the center x_0. For all $x \in J$ we have

$$|x - x_0| \le \frac{2}{3a(n+1)} \le \frac{1}{6a(n)} = \frac{\lambda(I) - \lambda(I')}{2},$$

hence $x \in I$ and $\|a(n)\,x\| \le a(n)\,|x - x_0| \le \dfrac{2a(n)}{3a(n+1)}$. □

Lemma 6. *Let $|z(n)| \ge 2$ and $a(n)/a(n+1) \le 1/4$. Then for every interval I such that $\lambda(I) = 4/(3a(n))$ there exists an interval $J \subseteq I$ such that $\lambda(J) = 4/(3a(n+1))$ and for all $x \in J$,*

$$\|a(n)\,x\| \le \frac{2a(n)}{3a(n+1)} + \frac{1}{|z(n)| - \frac{1}{2}} \quad \text{and} \quad \Big\|\sum_{j\le n} z(j)\,a(j)\,x\Big\| \ge \frac{1}{6}.$$

Proof. Let I' and x_0 be as in Lemma 5. Put $m = |\sum_{j\le n} z(j)\,a(j)|$. We have

$$\big(|z(n) - \tfrac{1}{2}\big)\,a(n) \le m \le \big(|z(n)| + \tfrac{1}{2}\big)\,a(n).$$

Since $\lambda(I') \ge 1/m$, there exists $x_1 \in I'$ such that $\|mx_1\| = 1/2$ and $|x_1 - x_0| \le 1/m$. Let J be an interval of the length $4/(3a(n+1))$ with the center x_1.

For all $x \in J$ we have

$$|x - x_1| \le \frac{2}{3a(n+1)} \le \frac{1}{6a(n)} = \frac{\lambda(I) - \lambda(I')}{2},$$

thus $x \in I$. Since also $2/(3a(n+1)) \le 1/(3m)$, we obtain

$$\|mx\| \ge \frac{1}{2} - m|x - x_1| \ge \frac{1}{6}.$$

We have $|x - x_0| \le |x - x_1| + |x_1 - x_0| \le 2/(3a(n+1)) + 1/m$, hence

$$\|a(n)\,x\| \le a(n)\,|x - x_0| \le \frac{2a(n)}{3a(n+1)} + \frac{1}{|z(n)| - \frac{1}{2}}.$$ □

Lemma 7. *Let $a(n)/a(n+1) \leq 1/8$. If I is an interval such that $\lambda(I) = 4/(3a(n))$ and for all $x \in I$, $\|\sum_{j<n} z(j)\, a(j)\, x\,| \geq 1/6$, then there exists an interval $J \subseteq I$ such that $\lambda(J) = 4/(3a(n+1))$ and for all $x \in J$,*

$$\|a(n)\, x\| \leq \frac{4a(n)}{3a(n+1)} \quad \textit{and} \quad \left\|\sum_{j\leq n} z(j)\, a(j)\, x\right\| \geq \frac{1}{6}.$$

Proof. Let I', x_0, m be as in Lemma 6. We have $\|mx_0\| = \|\sum_{j<n} z(j)\, a(j)\, x_0\| \geq 1/6$. Let J' be the longest interval containing x_0 on which the condition $\|mx\| \geq 1/6$ is satisfied. We have $\lambda(J') = 2/(3m) \geq 4/(3a(n+1))$, thus there exists an interval $J \subseteq J'$ of the length $\lambda(J) \geq 4/(3a(n+1))$ such that $x_0 \in J$. For all $x \in J$ we have

$$|x - x_0| \leq \frac{4}{3a(n+1)} \leq \frac{1}{6a(n)} = \frac{\lambda(I) - \lambda(I')}{2},$$

hence $x \in I$ and $\|a(n)\, x\| \leq a(n)\, |x - x_0| \leq \dfrac{4a(n)}{3a(n+1)}$. □

Lemma 8. *Let c, ε be reals such that $c \geq 0$ and $0 < \varepsilon \leq 1/24$. Let $z(n) \neq 0$, and let $a(n)/a(n+1) \leq 1/16$. If I is an interval such that $\lambda(I) = 4/(3a(n))$ and for all $x \in I$, $\|\sum_{j<n} z(j)\, a(j)\, x\| \geq c$, then there exists an interval $J \subseteq I$ such that $\lambda(J) = 4/(3a(n+1))$ and for all $x \in J$,*

$$a(n)\, x\| \leq \frac{4a(n)}{3a(n+1)} + 2\varepsilon \quad \textit{and} \quad \left\|\sum_{j\leq n} z(j)\, a(j)\, x\right\| \geq \min\left\{\frac{1}{6}, c + \varepsilon\right\}.$$

Proof. Let I', x_0, and m be as in Lemma 6. We have $m \geq a(n)/2$, and $\|mx_0\| = \|\sum_{j<n} z(j)\, a(j)\, x_0\| \geq c$. Let J' be an interval with the center x_0 such that $\lambda(J') = 2\varepsilon/m$.

If there exists $x_1 \in J'$ such that $mx_1\| \geq 1/6$, then we can find an interval J of the length $4/(3a(n+1))$ such that $x_1 \in J$ and for all $x \in J$, $\|mx\| \geq 1/6$. For $x \in J$ we obtain

$$|x - x_0| \leq |x - x_1| + |x_1 - x_0| \leq \frac{4}{3a(n+1)} + \frac{\varepsilon}{m} \leq \frac{1}{6a(n)} = \frac{\lambda(I) - \lambda(I')}{2},$$

hence $J \subseteq I$.

If $\|mx\| < 1/6$ all $x \in J'$, then there exists $x_1 \in \{x_0 - \varepsilon/m, x_0 + \varepsilon/m\}$ such that $\|mx_1\| = |mx_0\| + \varepsilon$. As in the previous case, there exists an interval J of the length $4/(3a(n+1))$ such that $x_1 \in J$ and for all $x \in J$, $\|mx\| \geq \|mx_1\| \geq c + \varepsilon$. Again $J \subseteq I$.

In both cases we obtain that for all $x \in J$, $\|mx\| \geq \min\{1/6, c + \varepsilon\}$, and

$$\|a(n)\, x\| \leq a(n)\, |x - x_0| \leq \frac{4a(n)}{3a(n+1)} + 2\varepsilon. \qquad \square$$

Proof of $A_a \subseteq A_b \rightarrow (2)$. We will show that if (2) is false, then there exists $x \in A_a \setminus A_b$. We will consider two cases.

(A) Let the set $\{|z_k(n)| : k, n \in \mathbb{N}\}$ be unbounded. Then there exist increasing sequences of natural numbers $\{n_i\}_{i\in\mathbb{N}}$, $\{k_i\}_{i\in\mathbb{N}}$ such that

(i) for all $n \geq n_0$, $a(n)/a(n+1) \leq 1/8$.

(ii) for all $i \in \mathbb{N}$, $|z_{k_i}(n_i)| \geq 2$,

(iii) $\lim_{i\to\infty} |z_{k_i}(n_i)| = \infty$,

(iv) for all $i \in \mathbb{N}$, and for all $n \geq n_{i+1}$, $z_{k_i}(n) = 0$.

We will define a sequence of intervals $\{I_n\}_{n\geq n_0}$ as follows. Take an arbitrary interval I_{n_0} such that $\lambda(I_{n_0}) = 4/(3a(n_0))$.

Let $n \geq n_0$ and let I_n be an interval of the length $4/(3a(n))$.

If $n = n_i$ for some i, then by Lemma 6 there exists an interval $I_{n+1} \subseteq I_n$ of the length $4/(3a(n+1))$ such that for all $x \in I_{n+1}$,

$$\|a(n)\,x\| \leq \frac{2a(n)}{3a(n+1)} + \frac{1}{|z_{k_i}(n_i)| - \frac{1}{2}} \quad \text{and} \quad \left\|\sum_{j\leq n} z_{k_i}(j)\,a(j)\,x\right\| \geq \frac{1}{6}.$$

Otherwise, $n_i < n < n_{i+1}$ for some i. We have $\left\|\sum_{j<n} z_{k_i}(j)\,a(j)\,x\right\| \geq 1/6$ for all $x \in I_n$. By Lemma 7 there exists an interval $I_{n+1} \subseteq I_n$ of the length $4/(3a(n+1))$ such that for all $x \in I_{n+1}$,

$$\|a(n)\,x\| \leq \frac{4a(n)}{3a(n+1)} \quad \text{and} \quad \left\|\sum_{j\leq n} z_{k_i}(j)\,a(j)\,x\right\| \geq \frac{1}{6}.$$

Let $x \in \bigcap_{n\geq n_0} I_n$. Since $\lim_{n\to\infty} \dfrac{a(n)}{a(n+1)} = 0$ and (iii), we have $x \in A_a$. For all $i \in \mathbb{N}$, the condition (iv) implies that

$$\|b(k_i)\,x\| = \left\|\sum_{j<n_{i+1}} z_{k_i}(j)\,a(j)\,x\right\| \geq \frac{1}{6},$$

since $x \in I_{n_{i+1}}$. Thus $x \notin A_b$.

(B) Let the set $\{s_k : k \in \mathbb{N}\}$ be unbounded, where $s_k = |\{n \in \mathbb{N} : z_k(n) \neq 0\}|$. Then there exist increasing sequences $\{n_i\}_{i\in\mathbb{N}}$ such that

(i) for all $n \geq n_0$, $a(n)/a(n+1) \leq 1/16$.

(ii) for all $i \in \mathbb{N}$, $s_{k_i} \geq n_i + i + 4$,

(iii) for all $i \in \mathbb{N}$, and for all $n \geq n_{i+1}$, $z_{k_i}(n) = 0$.

For $i \in \mathbb{N}$, let $m_i = |\{n \in \mathbb{N} : n \geq n_i \wedge z_{k_i}(n) \neq 0\}|$. From the condition (ii) it follows that $m_i \geq s_{k_i} - n_i \geq i + 4$, hence $\lim_{i\to\infty} m_i = \infty$. Put $\varepsilon_i = 1/(6m_i)$. We have $m_i \geq 4$, hence $\varepsilon_i \leq 1/24$.

As in the case (A), we will define a sequence of intervals $\{I_n\}_{n\geq n_0}$, starting with an arbitrary interval I_{n_0} such that $\lambda(I_{n_0}) = 4/(3a(n_0))$.

Let $n \geq n_0$ and let I_n be an interval of the length $4/(3a(n))$. Find $i \in \mathbb{N}$ such that $n_i \leq n < n_{i+1}$ and put

$$c_n = \min\left\{\left\|\sum_{j<n} z_{k_i}(j)\,a(j)\,x\right\| : x \in I_n\right\}.$$

If $z_{k_i}(n) = 0$, then by Lemma 5 there exists an interval $I_{n+1} \subseteq I_n$ of the length $4/(3a(n+1))$ such that for all $x \in I_{n+1}$, $\|a(n)\,x\| \leq 2a(n)/(3a(n+1))$. Clearly also $\|\sum_{j \leq n} z_{k_i}(j)\,a(j)\,x\| = \|\sum_{j<n} z_{k_i}(j)\,a(j)\,x\| \geq c_n$.

If $z_{k_i}(n) \neq 0$, then by Lemma 8 there exists an interval $I_{n+1} \subseteq I_n$ of the length $4/(3a(n+1))$ such that for all $x \in I_{n+1}$,

$$\|a(n)\,x\| \leq \frac{4a(n)}{3a(n+1)} + 2\varepsilon_i \quad \text{and} \quad \left\|\sum_{j \leq n} z_{k_i}(j)\,a(j)\,x\right\| \geq \min\left\{\frac{1}{6}, c_n + \varepsilon_i\right\}.$$

Let $x \in \bigcap_{n \geq n_0} I_n$. Since $\lim\limits_{n\to\infty} \dfrac{a(n)}{a(n+1)} = 0$ and $\lim\limits_{i\to\infty} \varepsilon_i = 0$, we have $x \in A_a$. For all $i \in \mathbb{N}$, the condition (iii) implies that

$$\|b(k_i)\,x\| = \left\|\sum_{j<n_{i+1}} z_{k_i}(j)\,a(j)\,x\right\| \geq \min\left\{\frac{1}{6}, m_i\varepsilon_i\right\} = \frac{1}{6},$$

since we have m_i-times increased the value $c_n \geq 0$ by ε_i. Hence $x \notin A_b$.

It is clear that if (2) is false, then either (A) or (B) is the case, and hence $A_a \subseteq A_b$ is false. □

Peroof of $A_a \subseteq A_b \to (1)$. We will show that if (1) is false, then there exists $x \in A_a \setminus A_b$. Again, we will consider two cases.

(A) Let us assume that there exist $t \in \mathbb{N}$ and an infinite set $K \subseteq \mathbb{N}$ such that for all $k \in K$, $z_k(t) \neq 0$, and for all $n > t$, the set $\{k \in K : z_k(n) \neq 0\}$ is finite. From Lemma 3 it follows that the set $\{z_k(n) : k \in \mathbb{N}\}$ is finite for every $n \leq t$, hence we can find integers $y(0), \ldots, y(t)$ and an infinite set $L \subseteq K$ such that for all $k \in L$ and for all $n \leq t$, $z_k(n) = y(n)$. Denote $m = \sum_{n \leq t} y(n)\,a(n)$. There exist increasing sequences of natural numbers $\{n_i\}_{i\in\mathbb{N}}$, $\{k_i\}_{i\in\mathbb{N}}$ such that

(i) $n_0 > t$,
(ii) for all $n \geq n_0$, $a(n)/a(n+1) \leq 1/8$,
(iii) for all $i \in \mathbb{N}$, $k_i \in L$,
(iv) for all $i, n \in \mathbb{N}$, if $z_{k_i}(n) \neq 0$, then $n \leq t$ or $n_i \leq n < n_{i+1}$.

If follows that for all $i \in \mathbb{N}$, $\sum_{j<n_i} z_{k_i}(j)\,a(j) = m$ and $\sum_{j<n_{i+1}} z_{k_i}(j)\,a(j) = b(k_i)$.

Let us define a sequence of intervals $\{I_n\}_{n\geq n_0}$ as follows. Take an arbitrary interval I of the length $2/(3|m|)$ such that for all $x \in I$, $\|mx\| \geq 1/6$. Since $|m| \leq a(t+1)/2$, we have $\lambda(I) \geq 4/(3a(t+1)) \geq 4/(3a(n_0))$, and thus there exists an interval $I_{n_0} \subseteq I$ of the length $4/(3a(n_0))$.

Let $n \geq n_0$ and let I_n be an interval of the length $4/(3a(n))$. Let $i \in \mathbb{N}$ be such that $n_i \leq n < n_{i+1}$. If $n = n_i$, then for all $x \in I_n$ we have $\|\sum_{j<n} z_{k_i}(j)\,a(j)\,x\| = \|mx\| \geq 1/6$. Hence by Lemma 7 there exists an interval $I_{n+1} \subseteq I_n$ of the length $4/3a(n+1)$ such that for all $x \in I_{n+1}$,

$$\|a(n)\,x\| \leq \frac{4a(n)}{3a(n+1)} \quad \text{and} \quad \left\|\sum_{j \leq n} z_{k_i}(j)\,a(j)\,x\right\| \geq \frac{1}{6}.$$

We can find such interval I_{n+1} for all n, $n_i \leq n < n_{i+1}$.

Let $x \in \bigcap_{n \geq n_0} I_n$. We have $\lim_{n\to\infty} \|a(n)\, x\| = 0$, thus $x \in A_a$. For every $i \in \mathbb{N}$ we obtain $\|b(k_i)\| = \left\|\sum_{j<n_{i+1}} z_{k_i}(j)\, a(j)\, x\right\| \geq 1/6$, and thus $x \notin A_b$.

(B) Let (A) be not the case, i.e. for every $t \in \mathbb{N}$ and for every infinite set $K \subseteq \{k \in \mathbb{N} : z_k(t) \neq 0\}$ there exists $n > t$ such that the set $\{k \in K : z_k(n) \neq 0\}$ is infinite. Then there exist increasing sequences of natural numbers $\{n_i\}_{i\in\mathbb{N}}$, $\{k_i\}_{i\in\mathbb{N}}$ such that

(i) for all $n \geq n_0$, $a(n)/a(n+1) \leq 1/8$,

(ii) for all $i, j \in \mathbb{N}$ and for all $n \leq \min\{n_i, n_j\}$, $z_{k_i}(n) = z_{k_j}(n)$,

(iii) for all $i \in \mathbb{N}$ and for all $n \in \mathbb{N}$ such that $n_0 \leq n \leq n_i$, $z_{k_i}(n) \neq 0$ if and only if $n = n_j$ for some $j \leq i$,

(iv) for all $i \in \mathbb{N}$ and for all $n \geq n_{i+1}$, $z_{k_i}(n) = 0$.

Let $m = \sum_{j<n_0} z_{k_0}(j)\, a(j)$.

We will define a sequence of intervals $\{I_n\}_{n\geq n_0}$ as follows. Let I be any interval of the length $2/(3|m|)$ such that for all $x \in I$, $\|mx\| \geq 1/6$. We have $|m| \leq a(n_0)/2$, hence there exists an interval $I_{n_0} \subseteq I$ of the length $4/(3a(n_0))$.

Let $n \geq n_0$ and let I_n be an interval of the length $4/(3a(n))$. Let $i \in \mathbb{N}$ be such that $n_i \leq n < n_{i+1}$. Let us assume that for all $x \in I_n$, $\left\|\sum_{j<n} z_{k_i}(j)\, a(j)\, x\right\| \geq 1/6$. This is clearly satisfied for $n = n_0$, for other n it will be proved by induction. By Lemma 7 there exists an interval $I_{n+1} \subseteq I_n$ of the length $4/(3a(n+1))$ such that for all $x \in I_{n+1}$,

$$\|a(n)\, x\| \leq \frac{4a(n)}{3a(n+1)} \quad \text{and} \quad \left\|\sum_{j\leq n} z_{k_i}(j)\, a(j)\, x\right\| \geq \frac{1}{6}.$$

We can do this for all n such that $n_i \leq n < n_{i+1}$. Since by the conditions (iii) and (ii), $z_{k_{i+1}}(n) = 0$ for all n such that $n_i < n < n_{i+1}$, and $z_{k_{i+1}}(n) = z_{k_i}(n)$ for all $n \leq n_i$, we obtain that for all $x \in I_{n_{i+1}}$,

$$\left\|\sum_{j<n_{i+1}} z_{k_{i+1}}(j)\, a(j)\, x\right\| = \left\|\sum_{j\leq n_i} z_{k_i}(j)\, a(j)\, x\right\| \geq \frac{1}{6}.$$

Moreover, from the condition (iv) it follows that $b(k_i) = \sum_{j<n_{i+1}} z_{k_i}(j)\, a(j)$, and thus for all $x \in I_{n_{i+1}}$, $\|b(k_i)\, x\| = \left\|\sum_{j<n_{i+1}} z_{k_i}(j)\, a(j)\, x\right\| \geq 1/6$.

Let $x \in \bigcap_{n\geq n_0} I_n$. We obtain $x \in A_a \setminus A_b$, and the proof of Theorem 4 is finished. □

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