A CLASSIFICATION OF TRIGONOMETRICAL THIN SETS AND THEIR INTERRELATIONS

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ABSTRACT. We introduce a uniform way of classifying thin sets of harmonic analysis related to absolute convergence of trigonometric series. This classification covers classical classes ($\mathcal{D}$, $\mathcal{PD}$, $\mathcal{A}$, $\mathcal{N}_0$, $\mathcal{N}$) and yields two new ones ($\mathcal{B}_0$ and $\mathcal{B}$). We study interrelations between these classes concerning combinatorial structure of thin sets.

In 1938 J. Marcinkiewicz [8] introduced the notion of N-set (in honour of V. V. Niemytzki): a set $X \subseteq [0,1]$ is an N-set iff there is a trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos 2\pi nx + b_n \sin 2\pi nx)$$

absolutely converging on X with $\sum_{n=1}^{\infty}(|a_n| + |b_n|) = \infty$ (i. e. not converging absolutely everywhere).

We can equivalently define the N-sets assuming that $a_n = 0$ for all n.

Modifying this definition we obtain a number of types of trigonometrical thin sets, depending on the form of terms in the series and on the convergence used. All but two of the considered classes are equal to the classes of D-, PD-, A-, N_0- and N-sets, known from the literature (e. g. [1], [3], [7]). We introduce the classes of B_0- and B-sets, which are new, although already implicitly considered.

It is known that all the classes mentioned above differ from each other (see e. g. [7]). In section 2 we generalize this fact. We get an estimation for the minimum size of a family of sets from one class which covers any set belonging to the other one.

Our notation follows that of [4]. Concerning historical notes we refer again to [4].

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1. Classification

Let $\{n_k\}_{k\in\omega}$ denote an increasing sequence of positive integers and $\{\varrho_k\}_{k\in\omega}$ a sequence of positive reals. We denote by $\|x\|$ the distance of a real x to the nearest integer. Since $2\|x\| \leq |\sin \pi x| \leq \pi\|x\|$, we can in our considerations replace $|\sin \pi x|$ and $\|x\|$ one by another.

Let us recall the definitions of classical thin sets, related to absolute convergence of trigonometric series:

Definition 1.1. *A set $X \subseteq [0,1]$ is*

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(1) *a D-set (a Dirichlet set) iff there exists a sequence* $\{n_k\}_{k\in\omega}$ *such that* $\|n_k x\|$ *converges uniformly to* 0 *on* X*,*
(2) *a PD-set (a pseudo-Dirichlet set) iff there exists a sequence* $\{n_k\}_{k\in\omega}$ *such that* $\|n_k x\|$ *converges quasinormally to* 0 *on* X*,*
(3) *an A-set (an Arbault set) iff there exists a sequence* $\{n_k\}_{k\in\omega}$ *such that* $\|n_k x\|$ *converges pointwise to* 0 *on* X*,*
(4) *an* N_0*-set iff there exists a sequence* $\{n_k\}_{k\in\omega}$ *such that* $\sum_{k=0}^{\infty}\|n_k x\| < \infty$ *for* $x \in X$*,*
(5) *an N-set iff there exist sequences* $\{\varrho_k\}_{k\in\omega}$*,* $\{n_k\}_{k\in\omega}$ *such that* $\sum_{k=0}^{\infty}\varrho_k = \infty$ *and* $\sum_{k=0}^{\infty}\varrho_k\|n_k x\| < \infty$ *for* $x \in X$*.*

Now define two new classes.

Definition 1.2. *A set* $X \subseteq [0,1]$ *is*

(6) *a* B_0*-set iff there exist a sequence* $\{n_k\}_{k\in\omega}$ *and a real constant* c *such that* $\sum_{k=0}^{\infty}\|n_k x\| \le c$ *for* $x \in X$*,*
(7) *a B-set iff there exist sequences* $\{\varrho_k\}_{k\in\omega}$*,* $\{n_k\}_{k\in\omega}$ *and a real constant* c *such that* $\sum_{k=0}^{\infty}\varrho_k = \infty$ *and* $\sum_{k=0}^{\infty}\varrho_k\|n_k x\| \le c$ *for* $x \in X$*.*

The classes of all D-, PD-, A-, N_0-, N-, B_0- and B-sets will be denoted by $\mathcal{D}$, $\mathcal{PD}$, $\mathcal{A}$, $\mathcal{N}_0$, $\mathcal{N}$, $\mathcal{B}_0$ and $\mathcal{B}$, respectively.

We can see that all definitions above have the same form. Only two parameters are changing: the type of convergence and the condition put on the coefficiets ϱ_k. We will study also some other possibilities for these parameters.

Let us consider the following conditions for the sequence $\{\varrho_k\}_{k\in\omega}$:

(i) $\sum_{k=0}^{\infty}\varrho_k = \infty$,
(ii) ϱ_k does not tend to 0,
(iii) ϱ_k tends to infinity.

Let us consider also the following types of convergence, where $\{f_k\}_{k\in\omega}$ is a sequence of non-negative functions defined on a set X:

(P) $\{f_k\}_{k\in\omega}$ converges pointwise to 0 on X;
(QN) $\{f_k\}_{k\in\omega}$ converges quasinormally to 0 on X;
(U) $\{f_k\}_{k\in\omega}$ converges uniformly to 0 on X;
(PS) $\sum_{k=0}^{\infty} f_k(x)$ converges pointwise on X, i. e. $\sum_{k=0}^{\infty} f_k(x) < \infty$ for $x \in X$;
(QNS) $\sum_{k=0}^{\infty} f_k(x)$ converges quasinormally on X, i. e. the sequence of its partial sums $\sum_{k=0}^{n} f_k(x)$ converges quasinormally on X;
(US) $\sum_{k=0}^{\infty} f_k(x)$ converges uniformly on X, i. e. the sequence of its partial sums $\sum_{k=0}^{n} f_k(x)$ converges uniformly on X;
(PNS) $\sum_{k=0}^{\infty} f_k(x)$ converges pseudonormally on X, i. e. there is a sequence of positive reals $\{\varepsilon_k\}_{k\in\omega}$ such that $\sum_{k=0}^{\infty}\varepsilon_k < \infty$ and $\forall x \in X\ \forall^\infty k\ f_k(x) \le \varepsilon_k$;
(NS) $\sum_{k=0}^{\infty} f_k(x)$ normally converges on X, i. e. there is a sequence of positive reals $\{\varepsilon_k\}_{k\in\omega}$ such that $\sum_{k=0}^{\infty}\varepsilon_k < \infty$ and $\forall x \in X\ \forall k\ f_k(x) \le \varepsilon_k$;
(BS) $\sum_{k=0}^{\infty} f_k(x)$ is bounded on X, i. e. there is a real c such that $\sum_{k=0}^{\infty} f_k(x) \le c$ for $x \in X$.

Let us note that the following implications hold true and that in general case no other one can be added:

$$\begin{tikzcd}
(\mathrm{NS}) \arrow[r] \arrow[d] & (\mathrm{PNS}) \arrow[d] & (\mathrm{BS}) \arrow[d] \\
(\mathrm{US}) \arrow[r] \arrow[d] \arrow[urr] & (\mathrm{QNS}) \arrow[r] \arrow[d] & (\mathrm{PS}) \arrow[d] \\
(\mathrm{U}) \arrow[r] & (\mathrm{QN}) \arrow[r] & (\mathrm{P})
\end{tikzcd}$$

Our aim is to examine all combinations of the properties (i)–(iii), (P)–(BS).

Definition 1.3. *Let* (a) *be one of the conditions* (i)–(iii) *and* (b) *one of the convergences* (P)–(BS). *Denote* $\mathcal{C}_b^a$ *the class of all sets* $X \subseteq [0,1]$ *for which there exists a sequence of the terms* $f_k(x) = \varrho_k \|n_k x\|$ *such that the sequences* $\{\varrho_k\}_{k\in\omega}$ *and* $\{f_k\}_{k\in\omega}$ *satisfy the conditions* (a) *and* (b).

It is easy to see that replacing the condition (ii) by $\liminf_{k\to\infty} \varrho_k > 0$ or $\forall k \in \omega\ \varrho_k = 1$ leaves the corresponding classes unchanged. Similarly we may replace the condition (iii) by $\limsup_{k\to\infty} \varrho_k = \infty$.

The following simple facts hold true:

Proposition 1.4. *Let* $\{h_k\}_{k\in\omega}$ *denote a sequence of non-negative functions and* $\{\sigma_k\}_{k\in\omega}$ *a sequence of positive reals.*

(1) *There exists a sequence* $\{\varrho_k\}_{k\in\omega}$ *such that* $\sum \varrho_k = \infty$ *and* $\lim \varrho_k = 0$. *If* $\lim \varrho_k = 0$ *and there is a real* c *such that* $\forall x \in X\ \forall k \in \omega\ h_k(x) \le c$ *then* $\varrho_k h_k(x)$ *converges uniformly to* 0 *on every set* X.

(2) *If* $\sum \varrho_k = \infty$ *(resp.* $\lim \varrho_k = \infty$*) then there exists a sequence* $\{\sigma_k\}_{k\in\omega}$ *such that* $\lim \sigma_k = 0$ *and* $\sum \sigma_k \varrho_k = \infty$ *(resp.* $\lim \sigma_k \varrho_k = \infty$*).*

(3) *If* $\varrho_k h_k(x)$ *converges pointwise to* 0 *on* X *and* $\lim \sigma_k = 0$ *then* $\sigma_k \varrho_k h_k(x)$ *converges quasinormally to* 0 *on* X.

(4) *If* $\sum \varrho_k h_k(x)$ *converges pointwise on* X *and* $\lim \sigma_k = 0$ *then* $\sum \sigma_k \varrho_k h_k(x)$ *converges quasinormally on* X.

(5) *If* $\sum \varrho_k h_k(x)$ *is bounded on* X *and* $\lim \sigma_k = 0$ *then* $\sum \sigma_k \varrho_k h_k(x)$ *converges uniformly on* X.

(6) *If* $\varrho_k h_k(x)$ *converges quasinormally (resp. uniformly) to* 0 *on* X *then there exists an increasing sequence* $\{k_j\}_{j\in\omega}$ *such that* $\sum \varrho_{k_j} h_{k_j}(x)$ *converges pseudonormally (resp. normally) on* X.

(7) *If* $\sum \varrho_k = \infty$ *and* $\sum \varrho_k h_k(x)$ *converges pseudonormally (resp. normally) on* X *then there exists an increasing sequence* $\{k_j\}_{j\in\omega}$ *such that* $h_{k_j}(x)$ *converges quasinormally (resp. uniformly) to* 0 *on* X.

(8) *If* $h_k(x)$ *converges quasinormally (resp. uniformly) to* 0 *on* X *then there exists a sequence* $\{\varrho_k\}_{k\in\omega}$ *such that* $\lim \varrho_k = \infty$ *and* $\varrho_k h_k(x)$ *converges quasinormally (resp. uniformly) to* 0 *on* X.

Put $h_k(x) = \|n_k x\|$. Then (1) implies that the classes $\mathcal{C}^{\mathrm{i}}_{\mathrm{P}}$, $\mathcal{C}^{\mathrm{i}}_{\mathrm{QN}}$, $\mathcal{C}^{\mathrm{i}}_{\mathrm{U}}$ consist of all subsets of $[0,1]$. ¿From (2) and (3) we get $\mathcal{C}^{\mathrm{iii}}_{\mathrm{P}} = \mathcal{C}^{\mathrm{iii}}_{\mathrm{QN}}$, from (2) and (4) $\mathcal{C}^{\mathrm{i}}_{\mathrm{PS}} = \mathcal{C}^{\mathrm{i}}_{\mathrm{QNS}}$, $\mathcal{C}^{\mathrm{iii}}_{\mathrm{PS}} = \mathcal{C}^{\mathrm{iii}}_{\mathrm{QNS}}$, from (2) and (5) $\mathcal{C}^{\mathrm{i}}_{\mathrm{BS}} = \mathcal{C}^{\mathrm{i}}_{\mathrm{US}}$, $\mathcal{C}^{\mathrm{iii}}_{\mathrm{BS}} = \mathcal{C}^{\mathrm{iii}}_{\mathrm{US}}$. ¿From (6) we can conclude that $\mathcal{C}^{\mathrm{ii}}_{\mathrm{QN}} = \mathcal{C}^{\mathrm{ii}}_{\mathrm{QNS}} = \mathcal{C}^{\mathrm{ii}}_{\mathrm{PNS}}$, $\mathcal{C}^{\mathrm{iii}}_{\mathrm{QN}} = \mathcal{C}^{\mathrm{iii}}_{\mathrm{QNS}} = \mathcal{C}^{\mathrm{iii}}_{\mathrm{PNS}}$, $\mathcal{C}^{\mathrm{ii}}_{\mathrm{U}} = \mathcal{C}^{\mathrm{ii}}_{\mathrm{US}} = \mathcal{C}^{\mathrm{ii}}_{\mathrm{NS}}$ and $\mathcal{C}^{\mathrm{iii}}_{\mathrm{U}} = \mathcal{C}^{\mathrm{iii}}_{\mathrm{US}} = \mathcal{C}^{\mathrm{iii}}_{\mathrm{NS}}$. The assertion (7) implies that $\mathcal{C}^{\mathrm{i}}_{\mathrm{PNS}} = \mathcal{C}^{\mathrm{ii}}_{\mathrm{QN}}$, $\mathcal{C}^{\mathrm{i}}_{\mathrm{NS}} = \mathcal{C}^{\mathrm{ii}}_{\mathrm{U}}$ and the assertion (8) that $\mathcal{C}^{\mathrm{ii}}_{\mathrm{QN}} = \mathcal{C}^{\mathrm{iii}}_{\mathrm{QN}}$, $\mathcal{C}^{\mathrm{ii}}_{\mathrm{U}} = \mathcal{C}^{\mathrm{iii}}_{\mathrm{U}}$.

Hence every class $\mathcal{C}_b^a$ is equal to one from Definitions 1.1, 1.2. In the following table *all* denotes the class of all subsets of $[0,1]$:

	(P)	(QN)	(U)	(PS)	(QNS)	(US)	(PNS)	(NS)	(BS)
(i)	*all*	*all*	*all*	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{B}$	$\mathcal{PD}$	$\mathcal{D}$	$\mathcal{B}$
(ii)	$\mathcal{A}$	$\mathcal{PD}$	$\mathcal{D}$	$\mathcal{N}_0$	$\mathcal{PD}$	$\mathcal{D}$	$\mathcal{PD}$	$\mathcal{D}$	$\mathcal{B}_0$
(iii)	$\mathcal{PD}$	$\mathcal{PD}$	$\mathcal{D}$	$\mathcal{PD}$	$\mathcal{PD}$	$\mathcal{D}$	$\mathcal{PD}$	$\mathcal{D}$	$\mathcal{D}$

The inclusions between these classes are the following ($\rightarrow$ means $\subseteq$):

$$\begin{array}{ccccc} & & \mathcal{A} & & \\ & & \uparrow & & \\ \mathcal{PD} & \rightarrow & \mathcal{N}_0 & \rightarrow & \mathcal{N} \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{D} & \rightarrow & \mathcal{B}_0 & \rightarrow & \mathcal{B} \end{array} \tag{D}$$

It is known that all these inclusions are proper and that no other one holds.

The following fact was proved by R. Salem [10] for N-sets. However, it holds for other classes too.

Proposition 1.5. *In Definition 1.3 we may replace* $\|n_k x\|$ *by* $\|\nu_k x + \alpha_k\|$*, where* $\{\nu_k\}_{k\in\omega}$ *is an unbounded sequence of positive reals,* $\{\alpha_k\}_{k\in\omega}$ *is a sequence of reals.*

Proof. Applying Proposition 1.4 to the functions $h_k(x) = \|\nu_k x + \alpha_k\|$ we obtain an identical table for classes defined using $\|\nu_k x + \alpha_k\|$ instead of $\|n_k x\|$. Therefore it is enough to consider the classes from Definitions 1.1 and 1.2. We give the proof only for case of A- and N-sets; for other classes it goes similarly.

(1) Suppose that $\lim_{k\to\infty} \|\nu_k + \alpha_k\| = 0$ for $x \in X$. There exists an increasing sequence $\{k_j\}_{j\in\omega}$ such that $\lim_{j\to\infty} \|\nu_{k_{j+1}} - \nu_{k_j}\| = 0$ and $\lim_{j\to\infty} \|\alpha_{k_{j+1}} - \alpha_{k_j}\| = 0$. Let n_j be the nearest integer to $\nu_{k_{j+1}} - \nu_{k_j}$. We may suppose that $\{n_j\}_{j\in\omega}$ is an increasing sequence of positive integers. For all $x \in [0,1]$ and $j \in \omega$ we have $\|n_j x\| \le \|\nu_{k_{j+1}} x + \alpha_{k_{j+1}}\| + \|\nu_{k_j} x + \alpha_{k_j}\| + \|\nu_{k_{j+1}} - \nu_{k_j}\| x + \|\alpha_{k_{j+1}} - \alpha_{k_j}\|$, and therefore $\lim_{j\to\infty} \|n_j x\| = 0$ for $x \in X$.

(2) Assume that $\sum_{k=0}^{\infty} \varrho_k = \infty$. There exists a sequence $\{s_k\}_{k\in\omega}$ of positive integers such that $\sum_{k=0}^{\infty} \varrho_k s_k^{-2} = \infty$, while $\sum_{k=0}^{\infty} \varrho_k s_k^{-3} < \infty$. For every $k \in \omega$ there is an integer p_k such that $1 \le p_k \le s_k^2$, $\|p_k \nu_k\| \le s_k^{-1}$ and $\|p_k \alpha_k\| \le s_k^{-1}$. Let n_k be the nearest integer to $p_k \nu_k$. We have $\|n_k x\| \le p_k \|\nu_k x + \alpha_k\| + \|p_k \nu_k\| x + \|p_k \alpha_k\|$ and hence $\varrho_k s_k^{-2} \|n_k x\| \le \varrho_k \|\nu_k x + \alpha_k\| + \varrho_k s_k^{-3}(x+1)$. We can see that $\sum_{k=0}^{\infty} \varrho_k s_k^{-2} \|n_k x\| < \infty$, whenever $\sum_{k=0}^{\infty} \varrho_k \|\nu_k x + \alpha_k\| < \infty$. □

2. Interrelations between the classes

Notice that all non-inclusions in the diagram (D) are based on the following four: $\mathcal{B}_0 \not\subseteq \mathcal{PD}$, $\mathcal{PD} \not\subseteq \mathcal{B}$, $\mathcal{B} \not\subseteq \mathcal{A}$ and $\mathcal{A} \not\subseteq \mathcal{N}$. Non-inclusion $\mathcal{PD} \not\subseteq \mathcal{B}$ is easy because B-sets are always nowhere dense, while there are PD-sets dense in $[0,1]$ (e. g. every countable set is a PD-set). In [7] S. Kahane gave examples of an N_0-set (actually B_0-set) which is not a countable union of PD-sets, an N-set (actually B-set) which is not a countable union of A-sets, and an A-set which is not a countable union of N-sets. Using the idea contained in his constructions we prove more informative facts about interrelations between these classes.

Let us start with some simple observations.

Lemma 2.1. *Let* $x \in [0,1]$*,* n *be a positive integer and* ε *be a real,* $0 \le \varepsilon \le \frac{1}{2}$*. Then the following conditions are equivalent:*

(1) $|\sin \pi n x| \leq \sin \pi\varepsilon$,
(2) $\|nx\| \leq \varepsilon$,
(3) $x \in \bigcup_{i=0}^{n-1} \left[\frac{i}{n}, \frac{i}{n} + \frac{\varepsilon}{n}\right] \cup \left[\frac{i+1}{n} - \frac{\varepsilon}{n}, \frac{i+1}{n}\right]$.

By an interval we mean a closed, nonempty subinterval of $[0,1]$. The length of an interval I we denote by $|I|$.

Corollary 2.2. *Let I be an interval, n be a positive integer, ε be a real, $0 \leq \varepsilon \leq \frac{1}{2}$.*

(1) *If $|I| \geq \frac{1}{n}$ then there exists an interval $J \subseteq I$ such that $\forall x \in J\ \|nx\| \leq \varepsilon$ and $|J| \geq \frac{\varepsilon}{n}$.*
(2) *If $|I| \geq \frac{2\varepsilon}{n}$ then there exists an interval $J \subseteq I$ such that $\forall x \in J\ \|nx\| \geq \varepsilon$ and $|J| \geq \min\left\{\frac{|I|}{2} - \frac{\varepsilon}{n}, \frac{1-2\varepsilon}{n}\right\}$.*

Lemma 2.3. *Let n, $n' \geq 2n$ be positive integers and I be an interval with $|I| \geq \frac{1}{n}$. Then there exists an interval $J \subseteq I$ such that $\forall x \in J\ \|nx\| \leq \frac{n}{n'}$ and $|J| \geq \frac{1}{n'}$.*

Proof. Take $\varepsilon = \frac{n}{n'} \leq \frac{1}{2}$ in Corollary 2.2 (1). $\square$

Let us note that T. Viola in his paper [12] using a similar fact proved that if $\sum_{k=0}^{\infty} \frac{n_k}{n_{k+1}} < \infty$ then the set of all x, such that the series $\sum_{k=0}^{\infty} \cos 2\pi n_k(x + \alpha_k)$ absolutely converges, is of the cardinality continuum.

Lemma 2.4. *Let n, n', m be positive integers and ε be a real such that $0 < \varepsilon \leq \frac{1}{16}$, $n \leq 2m < n'$ and $n \leq \varepsilon n'$. Let I be an interval with $|I| \geq \frac{1}{n}$. Then there exist intervals $J_0, J_1 \subseteq I$ with disjoint interiors, such that $\forall x \in J_0 \cup J_1\ \|nx\| \leq 8\varepsilon \wedge \|mx\| \geq \frac{\varepsilon}{2}$ and $|J_0|, |J_1| \geq \frac{1}{n'}$.*

Proof. By Corollary 2.2 (1) used for I, n and 8ε there is an interval $I' \subseteq I$ such that $\forall x \in I'\ \|nx\| \leq 8\varepsilon$ and $|I'| \geq \frac{8\varepsilon}{n}$. Let I'_0 and I'_1 be the left and the right half of I'. By Corollary 2.2 (2) used for I'_0, m and $\frac{\varepsilon}{2}$ there is $J_0 \subseteq I'_0$ such that $\forall x \in J_0\ \|mx\| \geq \frac{\varepsilon}{2}$ and

$$|J_0| \geq \min\left\{\frac{2\varepsilon}{n} - \frac{\varepsilon}{2m}, \frac{1-\varepsilon}{m}\right\} \geq \min\left\{\frac{2\varepsilon}{n} - \frac{\varepsilon}{n}, \frac{1}{2m}\right\} \geq \min\left\{\frac{\varepsilon}{n}, \frac{1}{n'}\right\} \geq \frac{1}{n'}.$$

Similarly there is $J_1 \subseteq I'_1$ such that $\forall x \in J_1\ \|mx\| \geq \frac{\varepsilon}{2}$ and $|J_1| \geq \frac{1}{n'}$. $\square$

We will need the following characterizations of two cardinal invariants of the ideal $\mathbb{K}$ of meager sets, due to J. Truss, A. W. Miller [9] and T. Bartoszyński [2]:

$$\operatorname{add}(\mathbb{K}) = \min\{\mathfrak{b}, \operatorname{cov}(\mathbb{K})\},$$
$$\operatorname{cov}(\mathbb{K}) = \min\{|X| : X \subseteq {}^{\omega}\omega \wedge \forall x \in {}^{\omega}\omega\ \exists y \in X\ \forall^{\infty} n \in \omega\ y(n) \neq x(n)\}.$$

Let us recall that

$$\mathfrak{b} = \min\{|X| : X \subseteq {}^{\omega}\omega \wedge \forall x \in {}^{\omega}\omega\ \exists y \in X\ \exists^{\infty} n \in \omega\ y(n) > x(n)\}.$$

For basic information see e. g. [5], [11].

Theorem 2.5. *Assume that $\sum_{k=0}^{\infty} \frac{n_k}{n_{k+1}} < \infty$. Let X be the N_0-set $\{x \in [0,1] : \sum_{k=0}^{\infty} \|n_k x\| < \infty\}$, $\mathcal{Y}$ be a family of PD-sets of size $|\mathcal{Y}| < \operatorname{add}(\mathbb{K})$, and I_0 be an interval with nonempty interior. Then the set $\left(X \setminus \bigcup \mathcal{Y}\right) \cap I_0$ has a perfect subset.*

Proof. Omitting finitely many terms from the sequence $\{n_k\}_{k\in\omega}$ we can ensure that $|I_0| \geq \frac{1}{n_0}$ and $n_{k+1} \geq 2n_k$ for all $k \in \omega$. Fix a sequence $\{\varepsilon_j\}_{j\in\omega}$ such that $0 < \varepsilon_j \leq \frac{1}{16}$ for all $j \in \omega$ and $\sum_{j=0}^{\infty} \varepsilon_j < \infty$.

Suppose that $\mathcal{Y} = \{Y_\xi : \xi < \kappa\}$, $\kappa < \operatorname{add}(\mathbb{K})$. For $\xi < \kappa$ let $\{m^\xi_l\}_{l\in\omega}$ be an increasing sequence of positive integers and $\{\theta^\xi_l\}_{l\in\omega}$ a sequence of positive reals such that $\lim_{l\to\infty}\theta^\xi_l = 0$ and $\forall x \in Y_\xi\ \forall^\infty l\ \|m^\xi_l x\| \le \theta^\xi_l$. For $m \in \omega$ let $k(m)$ be such that $n_{k(m)} \le 2m < n_{k(m)+1}$. For any $i, j \in \omega$ and $\xi < \kappa$ find an integer $l_\xi(i,j)$ such that $k\big(m^\xi_{l_\xi(i,j)}\big) \ge i$ and $\theta^\xi_{l_\xi(i,j)} < \frac{\varepsilon_j}{2}$. Put $p_\xi(i,j) = k\big(m^\xi_{l_\xi(i,j)}\big) + 1$.

Since $\kappa < \mathfrak{b}$, there exists $p \in {}^{\omega\times\omega}\omega$ such that $\forall \xi < \kappa\ \forall^\infty(i,j)\ p(i,j) \ge p_\xi(i,j)$, i. e. for every $\xi < \kappa$ there is $j_\xi \in \omega$ such that if $\min\{i,j\} \ge j_\xi$ then $p(i,j) \ge p_\xi(i,j)$. Put $p_0 = 0$, $p_{j+1} = p(p_j, j)$ for $j \in \omega$. Clearly if $j \ge j_\xi$ then $p_j \le k\big(m^\xi_{l_\xi(p_j,j)}\big) < p_\xi(p_j,j) \le p(p_j,j) = p_{j+1}$.

Since $\kappa < \operatorname{cov}(\mathbb{K})$, there exists a sequence $\{m_j\}_{j\in\omega}$ such that $\forall \xi < \kappa\ \exists^\infty j\ m_j = m^\xi_{l_\xi(p_j,j)}$. Denote $M_\xi = \big\{j \in \omega : j \ge j_\xi \wedge m_j = m^\xi_{l_\xi(p_j,j)}\big\}$, $M = \bigcup_{\xi<\kappa} M_\xi$ and $N = \{k(m_j) : j \in M\}$. The sets M_ξ, M, N are infinite and m is increasing on M.

Now let us construct our perfect set by an induction on k. Put $\mathcal{T}_0 = \{I_0\}$. Suppose that $\mathcal{T}_k$ is a family of intervals with pairwise disjoint interiors and that $|I| \ge \frac{1}{n_k}$ for every $I \in \mathcal{T}_k$.

If $k \in \omega\setminus N$ then $n_{k+1} \ge 2n_k$ and by Lemma 2.3 for any $I \in \mathcal{T}_k$ there exists $J^I \subseteq I$ such that $\forall x \in J^I\ \|n_k x\| \le \frac{n_k}{n_{k+1}}$ and $|J^I| \ge \frac{1}{n_{k+1}}$. Put $\mathcal{T}_{k+1} = \{J^I : I \in \mathcal{T}_k\}$.

If $k \in N$ then $k = k(m_j)$ for some $j \in M$, $n_k \le 2m_j < n_{k+1}$ and $n_k \le \varepsilon_j n_{k+1}$. By Lemma 2.4 for any $I \in \mathcal{T}_k$ there exist $J^I_0, J^I_1 \subseteq I$ such that $\forall x \in J^I_0 \cup J^I_1\ \|n_k x\| \le 8\varepsilon_j \wedge \|m_j x\| \ge \frac{\varepsilon_j}{2}$ and $|J^I_0|, |J^I_1| \ge \frac{1}{n_{k+1}}$. Put $\mathcal{T}_{k+1} = \{J^I_0, J^I_1 : I \in \mathcal{T}_k\}$.

The set $P = \bigcap_{k\in\omega}\bigcup \mathcal{T}_k$ is perfect subset of I_0. If $x \in P$ then $\sum_{k=0}^\infty \|n_k x\| = \sum_{k\in\omega\setminus N}\|n_k x\| + \sum_{j\in M}\|n_{k(m_j)}x\| \le \sum_{k\in\omega\setminus N}\frac{n_k}{n_{k+1}} + \sum_{j\in M} 8\varepsilon_j < \infty$, and hence $x \in X$. Moreover, for any $\xi < \kappa$ and $j \in M_\xi$ we have $m_j = m^\xi_{l_\xi(p_j,j)}$, $\|m_j x\| \ge \frac{\varepsilon_j}{2} > \theta^\xi_{l_\xi(p_j,j)}$, and therefore $x \notin Y_\xi$. Hence $P \subseteq X \setminus \bigcup\mathcal{Y}$. □

Theorem 2.6. *Assume that* $\sum_{k=0}^\infty \frac{n_k}{n_{k+1}} < \infty$. *Let* X *be the* B_0*-set* $\{x \in [0,1] : \sum_{k=0}^\infty \|n_k x\| \le c\}$, *where* c *is a sufficiently big real, and* $\mathcal{Y}$ *be a family of PD-sets of size* $|\mathcal{Y}| < \operatorname{add}(\mathbb{K})$. *Then the set* $X \setminus \bigcup\mathcal{Y}$ *has a perfect subset.*

Proof. Find k_0 such that $n_{k+1} \ge 2n_k$ for $k \ge k_0$ and take $c > \frac{k_0}{2} + \sum_{k=k_0}^\infty \frac{n_k}{n_{k+1}}$. Put $I_0 = [0,1]$ and choose $\{\varepsilon_j\}_{j\in\omega}$ such that $\frac{k_0}{2} + \sum_{k=k_0}^\infty \frac{n_k}{n_{k+1}} + 8\sum_{j=0}^\infty \varepsilon_j \le c$. Similarly as in Theorem 2.5 there is a perfect set $P \subseteq I_0$ disjoint with every $Y \in \mathcal{Y}$, such that if $x \in P$ then $\sum_{k=k_0}^\infty \|n_k x\| \le \sum_{k=k_0}^\infty \frac{n_k}{n_{k+1}} + 8\sum_{j=0}^\infty \varepsilon_j$, and hence $P \subseteq X$. □

Theorem 2.7. *Assume that* $\lim_{k\to\infty}\varrho_k = 0$, $\sum_{k=0}^\infty \varrho_k = \infty$ *and* $\sum_{k=0}^\infty \varrho_k \frac{n_k}{n_{k+1}} < \infty$. *Let* X *be the N-set* $\{x \in [0,1] : \sum_{k=0}^\infty \varrho_k\|n_k x\| < \infty\}$, $\mathcal{Y}$ *be a family of A-sets of size* $|\mathcal{Y}| < \operatorname{add}(\mathbb{K})$ *and* I_0 *be an interval with nonempty interior. Then the set* $\big(X \setminus \bigcup\mathcal{Y}\big) \cap I_0$ *has a perfect subset.*

Proof. Fix a real ε, $0 < \varepsilon \le \frac{1}{16}$, and a sequence $\{\theta_j\}_{j\in\omega}$ of positive reals such that $\sum_{j=0}^\infty \theta_j < \infty$. Denote $S = \big\{k \in \omega : |I_0| \ge \frac{1}{n_k} \wedge n_k \le \varepsilon n_{k+1}\big\}$. Clearly $\sum_{k\in\omega\setminus S}\varrho_k < \infty$, and hence S is infinite. Denote $S(k)$ the k-th member of S.

Suppose that $\mathcal{Y} = \{Y_\xi : \xi < \kappa\}$, $\kappa < \operatorname{add}(\mathbb{K})$. For $\xi < \kappa$ let $\{m^\xi_l\}_{l\in\omega}$ be an increasing sequence of positive integers such that $\forall x \in Y_\xi\ \lim_{l\to\infty}\|m^\xi_l x\| = 0$. For $m \in \omega$ let $k(m)$ be such that $n_{S(k(m))} \le 2m < n_{S(k(m)+1)}$. For any $i, j \in \omega$ and $\xi < \kappa$ find an integer $l_\xi(i,j)$ such that $k\big(m^\xi_{l_\xi(i,j)}\big) \ge i$ and $\varrho_{S(k(m^\xi_{l_\xi(i,j)}))} \le \theta_j$.

Similarly as in Theorem 2.5 we can find an increasing sequence $\{p_j\}_{j\in\omega}$, a sequence $\{m_j\}_{j\in\omega}$ and for any $\xi<\kappa$ an infinite set $M_\xi\subseteq\omega$, such that if $\xi<\kappa$ and $j\in M_\xi$ then $m_j=m^\xi_{l_\xi(p_j,j)}$, $p_j\le k(m_j)<p_{j+1}$ and $\varrho_{S(k(m_j))}\le\theta_j$. Denote $M=\bigcup_{\xi<\kappa}M_\xi$ and $N=\{k(m_j):j\in M\}$. Clearly m is increasing on M and M, N are infinite.

We know that $|I_0|\ge\frac{1}{n_{S(0)}}$ and $n_{S(k)}\le\varepsilon n_{S(k)+1}$ for all $k\in\omega$. By a similar construction as in Theorem 2.5 we can construct a perfect set $P\subseteq I_0$ such that if $x\in P$ then $\|n_{S(k)}x\|\le\frac{n_{S(k)}}{n_{S(k)+1}}$ for $k\in\omega\setminus N$, $\|n_{S(k)}x\|\le 8\varepsilon\wedge\|m_jx\|\ge\frac{\varepsilon}{2}$ for $k=k(m_j)$, $j\in M$.

For $x\in P$ holds $\sum_{k=0}^\infty\varrho_k\|n_kx\|\le\frac12\sum_{k\in\omega\setminus S}\varrho_k+\sum_{k\in S}\varrho_k\|n_kx\|<\infty$, since $\sum_{k\in S}\varrho_k\|n_kx\|=\sum_{k\in\omega\setminus N}\varrho_{S(k)}\|n_{S(k)}x\|+\sum_{j\in M}\varrho_{S(k(m_j))}\|n_{S(k(m_j))}x\|\le\sum_{k\in\omega\setminus N}\varrho_{S(k)}\frac{n_{S(k)}}{n_{S(k)+1}}+\sum_{j\in M}8\varepsilon\theta_j\le\sum_{k\in S}\varrho_k\frac{n_k}{n_{k+1}}+8\varepsilon\sum_{j=0}^\infty\theta_j$, and therefore $x\in X$. Moreover, for $\xi<\kappa$ and $j\in M_\xi$ we have $\|m^\xi_{l_\xi(p_j,j)}x\|=\|m_jx\|\ge\frac{\varepsilon}{2}$, and hence $x\notin Y_\xi$. We get $P\subseteq X\setminus\bigcup\mathcal{Y}$. □

Theorem 2.8. *Assume the hypotheses of Theorem 2.7. Let X be the B-set $\{x\in[0,1]:\sum_{k=0}^\infty\varrho_k\|n_kx\|\le c\}$, where c is a sufficiently big real, and $\mathcal{Y}$ be a family of A-sets of size $|\mathcal{Y}|<\operatorname{add}(\mathbb{K})$. Then the set $X\setminus\bigcup\mathcal{Y}$ has a perfect subset.*

Proof. Fix a real ε, $0<\varepsilon\le\frac{1}{16}$, and denote $S=\{k\in\omega:n_k\le\varepsilon n_{k+1}\}$. It is sufficient to take $c>\frac12\sum_{k\in\omega\setminus S}\varrho_k+\sum_{k\in S}\varrho_k\frac{n_k}{n_{k+1}}$. Put $I_0=[0,1]$ and choose $\{\theta_j\}_{j\in\omega}$ such that $\frac12\sum_{k\in\omega\setminus S}\varrho_k+\sum_{k\in S}\varrho_k\frac{n_k}{n_{k+1}}+8\varepsilon\sum_{j=0}^\infty\theta_j\le c$. Similarly as in Theorem 2.7 there exists a perfect set $P\subseteq I_0$, disjoint with every $Y\in\mathcal{Y}$, such that if $x\in P$ then $\sum_{k=0}^\infty\varrho_k\|n_kx\|\le\frac12\sum_{k\in\omega\setminus S}\varrho_k+\sum_{k\in S}\varrho_k\frac{n_k}{n_{k+1}}+8\varepsilon\sum_{j=0}^\infty\theta_j\le c$. □

The following lemma appears in [7] in a slightly different form.

Lemma 2.9. *Let $\{\varrho_n\}_{n\in\omega}$ be a sequence of positive reals, a, b, θ be positive reals and I be an interval such that $a<b$, $\theta\le\frac{1}{14}$ and $|I|\ge\frac{6\theta}{a}$. Then there exists an interval $J\subseteq I$ such that $\forall x\in J$ $\sum_{a\le n<b}\varrho_n\|nx\|\ge\frac{\theta}{4}\sum_{a\le n<b}\varrho_n$ and $|J|\ge\frac{2\theta}{b}$.*

Proof. Suppose for now that $b\le 3a$. There exists $\bar x$ such that $\left[\bar x-\frac{\theta}{b},\bar x+\frac{3\theta}{a}+\frac{2\theta}{b}\right]\subseteq I$. Denote $A=\{n\in\omega:a\le n<b\wedge\|n\bar x\|\ge2\theta\}$, $B=\{n\in\omega:a\le n<b\wedge\|n\bar x\|<2\theta\}$.

If $\sum_{n\in A}\varrho_n\ge\frac12\sum_{n\in A\cup B}\varrho_n$, put $J=\left[\bar x-\frac{\theta}{b},\bar x+\frac{\theta}{b}\right]$. Then for $x\in J$ and $n\in A$ we have $\|nx\|\ge\|n\bar x\|-\|n(x-\bar x)\|\ge2\theta-n|x-\bar x|\ge2\theta-\theta=\theta$.

Otherwise, if $\sum_{n\in B}\varrho_n>\frac12\sum_{n\in A\cup B}\varrho_n$, put $J=\left[\bar x+\frac{3\theta}{a},\bar x+\frac{3\theta}{a}+\frac{2\theta}{b}\right]$. Then for $x\in J$ and $n\in B$ we have $\frac{3\theta}{a}\le|x-\bar x|\le\frac{3\theta}{a}+\frac{2\theta}{b}\le\frac{11\theta}{b}$, $3\theta\le n|x-\bar x|\le11\theta\le1-3\theta$, and hence $\|nx\|\ge\|n(x-\bar x)\|-\|n\bar x\|\ge3\theta-2\theta=\theta$.

In both cases we get $\sum_{a\le n<b}\varrho_n\ge\frac{\theta}{2}\sum_{a\le n<b}\varrho_n$.

In case that $b>3a$ put $a_0=a$, $a_1=3a_0$, $a_2=3a_1,\dots$, $a_k=b\le3a_{k-1}$. Denote $A=\{n\in\omega:a_i\le n<a_{i+1}\text{ for }i\text{ even}\}$, $B=\{n\in\omega:a_i\le n<a_{i+1}\text{ for }i\text{ odd}\}$.

If $\sum_{n\in A}\varrho_n\ge\frac12\sum_{n\in A\cup B}\varrho_n$ then iterating the previous consideration we can find intervals $I\supseteq I_0\supseteq I_2\supseteq\dots$ such that for i even holds

$$(*)\qquad |I_i|\ge\frac{2\theta}{a_{i+1}}=\frac{6\theta}{a_{i+2}}\ \text{ and }\ \forall x\in I_i\ \sum_{a_i\le n<a_{i+1}}\varrho_n\|nx\|\ge\frac{\theta}{2}\sum_{a_i\le n<a_{i+1}}\varrho_n.$$

Similarly, if $\sum_{n\in B}\varrho_n>\frac12\sum_{n\in A\cup B}\varrho_n$ then there exist intervals $I\supseteq I_1\supseteq I_3\supseteq\dots$ such that $(*)$ holds for i odd. In both cases let J be the last interval (I_{k-2} or I_{k-1}). We get $|J|\ge\frac{2\theta}{b}$ and $\forall x\in J$ $\sum_{a\le n<b}\varrho_n\|nx\|\ge\frac{\theta}{4}\sum_{a\le n<b}\varrho_n$. □

Theorem 2.10. *Assume that* $\lim_{k\to\infty} \frac{n_k}{n_{k+1}} = 0$. *Let* X *be the A-set* $\{x \in [0,1] : \lim_{k\to\infty} \|n_k x\| = 0\}$, $\mathcal{Y}$ *be a family of N-sets of size* $|\mathcal{Y}| < \operatorname{add}(\mathbb{K})$ *and* I_0 *be an interval with nonempty interior. Then the set* $\big(X \setminus \bigcup \mathcal{Y}\big) \cap I_0$ *has a perfect subset.*

Proof. Omitting finitely many terms from the sequnce $\{n_k\}_{k\in\omega}$ we can ensure that $|I_0| \geq \frac{1}{n_0}$ and $n_{k+1} \geq 2n_k$ for all $k \in \omega$. Fix a sequence $\{\varepsilon_j\}_{j\in\omega}$ such that $0 < \varepsilon_j \leq \frac{1}{2}$ and $\lim_{j\to\infty} \varepsilon_j = 0$.

Suppose that $\mathcal{Y} = \{Y_\xi : \xi < \kappa\}$, $\kappa < \operatorname{add}(\mathbb{K})$. For $\xi < \kappa$ let $\{\varrho^\xi_n\}_{n\in\omega}$ be a sequence of positive rationals such that $\sum_{n=0}^\infty \varrho^\xi_n = \infty$ and $\forall x \in Y_\xi$ $\sum_{n=0}^\infty \varrho^\xi_n \|nx\| < \infty$. Denote $c = \frac{1}{14}$ and fix $\xi < \kappa$. There are two possibilities.

(a) There exists $d > c$ such that

$$\sum_{k=0}^{\infty} \sum_{cn_k \leq n < dn_k} \varrho^\xi_n = \infty.$$

Then we can choose θ^ξ_k such that $\theta^\xi_k \geq d\frac{n_k}{n_{k+1}}$, $\lim_{k\to\infty} \theta^\xi_k = 0$ and

$$\sum_{k=0}^{\infty} \theta^\xi_k \sum_{cn_k \leq n < dn_k} \varrho^\xi_n = \infty.$$

Denote $a^\xi_k = cn_k$, $b^\xi_k = dn_k$ for all $k \in \omega$.

(b) For all $d > c$ holds

$$\sum_{k=0}^{\infty} \sum_{cn_k \leq n < dn_k} \varrho^\xi_n < \infty.$$

Then for all $d > c$ we have

$$\sum_{k=0}^{\infty} \sum_{dn_k \leq n < cn_{k+1}} \varrho^\xi_n = \infty$$

and hence we can choose d_k such that $c < d_k < c\frac{n_{k+1}}{n_k}$, $\lim_{k\to\infty} d_k = \infty$ and

$$\sum_{k=0}^{\infty} \sum_{d_k n_k \leq n < cn_{k+1}} \varrho^\xi_n = \infty.$$

Denote $a^\xi_k = d_k n_k$, $b^\xi_k = cn_{k+1}$ and $\theta^\xi_k = c$ for all $k \in \omega$.

In both cases we have found sequences $\{a^\xi_k\}_{k\in\omega}$, $\{b^\xi_k\}_{k\in\omega}$ and $\{\theta^\xi_k\}_{k\in\omega}$ such that $\sum_{k=0}^\infty \theta^\xi_k \sum_{a^\xi_k \leq n < b^\xi_k} \varrho^\xi_n = \infty$. Moreover, for every $j \in \omega$ there exists $k^\xi_j \in \omega$ such that for all $k \geq k^\xi_j$ holds

$$\frac{\varepsilon_j}{n_k} \geq \frac{6\theta^\xi_k}{a^\xi_k}, \ \frac{2\theta^\xi_k}{b^\xi_k} \geq \frac{2}{n_{k+1}}, \ \theta^\xi_k \leq \frac{1}{14} \ \text{ and } \ a^\xi_k < b^\xi_k \leq a^\xi_{k+1}.$$

Let us note that we can choose a^ξ_k, b^ξ_k and θ^ξ_k rational.

For any $i, j \in \omega$ and $\xi < \kappa$ find integers $r_\xi(i,j)$, $s_\xi(i,j)$ and $p_\xi(i,j)$ such that

$$k^\xi_j \leq r_\xi(i,j) < s_\xi(i,j), \ \big[r_\xi(i,j), s_\xi(i,j)\big) \subseteq \big[i, p_\xi(i,j)\big),$$
$$\bigcup_{r_\xi(i,j) \leq k < s_\xi(i,j)} \big[a^\xi_k, b^\xi_k\big) \subseteq \big[n_i, n_{p_\xi(i,j)}\big) \ \text{ and } \sum_{r_\xi(i,j) \leq k < s_\xi(i,j)} \theta^\xi_k \sum_{a^\xi_k \leq n < b^\xi_k} \varrho^\xi_n \geq 1.$$

Since $\kappa < \mathfrak{b}$, there exists $p \in {}^{\omega\times\omega}\omega$ such that $\forall\xi < \kappa\ \forall^\infty(i,j)\ p(i,j) \geq p_\xi(i,j)$, i. e. for every $\xi < \kappa$ there is $j_\xi \in \omega$ such that if $\min\{i,j\} \geq j_\xi$ then $p(i,j) \geq p_\xi(i,j)$. Put $p_0 = 0$, $p_{j+1} = p(p_j, j)$ for $j \in \omega$ and denote $r^\xi_j = r_\xi(p_j, j)$, $s^\xi_j = s_\xi(p_j, j)$. We can see that if $j \geq j_\xi$ then $[r^\xi_j, s^\xi_j) \subseteq [p_j, p_{j+1})$, $\bigcup_{r^\xi_j \leq k < s^\xi_j} [a^\xi_k, b^\xi_k) \subseteq [n_{p_j}, n_{p_{j+1}})$ and $\sum_{r^\xi_j \leq k < s^\xi_j} \theta^\xi_k \sum_{a^\xi_k \leq n < b^\xi_k} \varrho^\xi_n \geq 1$.

For any $j \in \omega$, $\xi < \kappa$, we can code $\langle r^\xi_j, s^\xi_j, \langle a^\xi_k, b^\xi_k, \theta^\xi_k\rangle_{p_j \leq k < p_{j+1}}, \langle\varrho^\xi_n\rangle_{n_{p_j} \leq n < n_{p_{j+1}}}\rangle$ by $m^\xi_j \in \omega$. Since $\kappa < \mathrm{cov}(\mathbb{K})$, there exists a sequence $\{m_j\}_{j\in\omega}$ such that $\forall\xi < \kappa\ \exists^\infty j\ m_j = m^\xi_j$. If $m_j = m^\xi_j$, let $r_j = r^\xi_j$, $s_j = s^\xi_j$, $a_k = a^\xi_k$, $b_k = b^\xi_k$, $\theta_k = \theta^\xi_k$ for $p_j \leq k < p_{j+1}$, and $\varrho_n = \varrho^\xi_n$ for $n_{p_j} \leq n < n_{p_{j+1}}$. Denote $M_\xi = \{j \in \omega : j \geq j_\xi \wedge m_j = m^\xi_j\}$ for $\xi < \kappa$, $M = \bigcup_{\xi<\kappa} M_\xi$ and $N = \omega \cap \bigcup_{j\in M} [r_j, s_j)$.

Now let us construct the perfect set P by an induction on k. Put $\mathcal{T}_0 = \{I_0\}$. Suppose that $\mathcal{T}_k$ is a family of intervals with pairwise disjoint interiors and that $|I| \geq \frac{1}{n_k}$ for every $I \in \mathcal{T}_k$.

If $k \in \omega\setminus N$ then $n_{k+1} \geq 2n_k$ and by Lemma 2.3 for any $I \in \mathcal{T}_k$ there exists $J^I \subseteq I$ such that $\forall x \in J^I\ \|n_k x\| \leq \frac{n_k}{n_{k+1}}$ and $|J^I| \geq \frac{1}{n_{k+1}}$. Put $\mathcal{T}_{k+1} = \{J^I : I \in \mathcal{T}_k\}$.

If $k \in N$ then $r_j \leq k < s_j$ for some $j \in M$. By Corollary 2.2 (1) there exists $J^I \subseteq I$ such that $\forall x \in J^I\ \|n_k x\| \leq \varepsilon_j$ and $|J^I| \geq \frac{\varepsilon_j}{n_k}$. Since $a_k < b_k$, $\theta_k \leq \frac{1}{14}$ and $\frac{\varepsilon_j}{n_k} \geq \frac{6\theta_k}{a_k}$, by Lemma 2.9 there exists $\bar{J}^I \subseteq J^I$ such that

$$\forall x \in \bar{J}^I \sum_{a_k \leq n < b_k} \varrho_n \|nx\| \geq \frac{\theta_k}{4} \sum_{a_k \leq n < b_k} \varrho_n \quad\text{and}\quad |\bar{J}^I| \geq \frac{2\theta_k}{b_k} \geq \frac{2}{n_{k+1}}.$$

Denote $\bar{J}^I_0$ and $\bar{J}^I_1$ the left and the right half of $\bar{J}^I$ and put $\mathcal{T}_{k+1} = \{\bar{J}^I_0, \bar{J}^I_1 : I \in \mathcal{T}_k\}$.

Since N is infinite, the set $P = \bigcap_{k\in\omega} \bigcup \mathcal{T}_k$ is a perfect subset of I_0. For all $x \in P$ we have $\lim_{k\to\infty} \|n_k x\| = 0$, and hence $P \subseteq X$. Moreover, if $\xi < \kappa$ and $j \in M_\xi$ then

$$\sum_{r^\xi_j \leq k < s^\xi_j} \sum_{a^\xi_k \leq n < b^\xi_k} \varrho^\xi_n \|nx\| = \sum_{r_j \leq k < s_j} \sum_{a_k \leq n < b_k} \varrho_n \|nx\| \geq$$
$$\sum_{r_j \leq k < s_j} \frac{\theta_k}{4} \sum_{a_k \leq n < b_k} \varrho_n = \sum_{r^\xi_j \leq k < s^\xi_j} \frac{\theta_k}{4} \sum_{a^\xi_k \leq n < b^\xi_k} \varrho^\xi_n \geq \frac{1}{4},$$

and hence $\sum_{n=0}^\infty \varrho^\xi_n \|nx\| \geq \sum_{j\in M_\xi} \sum_{r^\xi_j \leq k < s^\xi_j} \sum_{a^\xi_k \leq n < b^\xi_k} \varrho^\xi_n \|nx\| = \infty$, i. e. $x \notin Y_\xi$. We get $P \subseteq X \setminus \bigcup \mathcal{Y}$. $\square$

It is easy to see that for any increasing sequence $\{n_k\}_{k\in\omega}$ of positive integers there is a subsequence $\{n_{k_j}\}_{j\in\omega}$ such that $\lim_{j\to\infty} \frac{n_{k_j}}{n_{k_{j+1}}} = 0$, or $\sum_{j=0}^\infty \frac{n_{k_j}}{n_{k_{j+1}}} < \infty$. Hence the family $\mathcal{F}$ of all sets X satisfying the conditions of Theorem 2.5 (resp. 2.6, 2.10) is a basis for the class $\mathcal{N}_0$ (resp. $\mathcal{B}_0$, $\mathcal{A}$), i. e. every N_0-set (resp. B_0-set, A-set) is included in a set $X \in \mathcal{F}$. As immediate consequences we get the following:

Corollary 2.11. *For any N_0-set X there is N_0-set $X' \supseteq X$ such that for any family $\mathcal{Y}$ of PD-sets of size $|\mathcal{Y}| < \mathrm{add}(\mathbb{K})$ and for any interval $I_0 \subseteq [0,1]$ with nonempty interior, the set $(X' \setminus \bigcup \mathcal{Y}) \cap I_0$ has a perfect subset.*

Corollary 2.12. *For any B_0-set X there is B_0-set $X' \supseteq X$ such that for any family $\mathcal{Y}$ of PD-sets of size $|\mathcal{Y}| < \mathrm{add}(\mathbb{K})$, the set $X' \setminus \bigcup \mathcal{Y}$ has a perfect subset.*

Corollary 2.13. *For any A-set X there is A-set $X' \supseteq X$ such that for any family $\mathcal{Y}$ of N-sets of size $|\mathcal{Y}| < \mathrm{add}(\mathbb{K})$ and for any interval $I_0 \subseteq [0,1]$ with nonempty interior, the set $\big(X' \setminus \bigcup \mathcal{Y}\big) \cap I_0$ has a perfect subset.*

We do not know whether a similar result holds true in case of N-sets (resp. B-sets). The following question is open:

Question 2.14. *Is it true that for any N-set X there are sequences $\{\varrho_k\}_{k\in\omega}$ and $\{n_k\}_{k\in\omega}$ such that $\sum_{k=0}^{\infty} \varrho_k = \infty$, $\sum_{k=0}^{\infty} \varrho_k \frac{n_k}{n_{k+1}} < \infty$ and $\sum_{k=0}^{\infty} \varrho_k \|n_k x\| < \infty$ for all $x \in X$?*

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