ARBAULT PERMITTED SETS ARE PERFECTLY MEAGER

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Abstract. We prove that every set permitted for the family of Arbault sets is perfectly meager. This negatively answers a question whether the existence of permitted sets of cardinality continuum is provable in ZFC.

A set $X \subseteq \mathbb{R}$ is called an *Arbault set* if there exists an increasing sequence of natural numbers $\{n_k\}_{k\in\mathbb{N}}$ such that for all $x \in X$,

$$\lim_{k\to\infty} \sin \pi n_k x = 0.$$

J. Arbault studied sets of this kind in his paper [1]. It is well known (see e.g. [2]) that these sets are meager and have Lebesgue measure zero. From the definition it immediately follows that Arbault sets are closed under taking subsets. However, they do not form an ideal since the union of two Arbault sets need not to be an Arbault set. It was proved by N. N. Kholshchevnikova that the union of an Arbault set and a countable set is an Arbault set, see [2].

Let $\mathcal{A}$ denote the family of all Arbault sets. We say that a set X is *permitted for the family* $\mathcal{A}$ (or *Arbault permitted*, *A-permitted*) if for every $Y \in \mathcal{A}$, $X \cup Y \in \mathcal{A}$.

In [2] it was proved that every γ-set of reals is A-permitted. By [6], if $\mathfrak{p} = \mathfrak{c}$ then there exists a γ-set of size $\mathfrak{c}$. Hence is consistent with ZFC that there exists an A-permitted set having cardinality of continuum. Some improvements of these results were proved in [9]. However, it was not known whether the existence of A-permitted sets of size $\mathfrak{c}$ is provable in ZFC. In this paper we show that the answer is negative.

We will prove that every set permitted for the family $\mathcal{A}$ is perfectly meager. A subset X of a topological space is *perfectly meager* if for every perfect set P, X is meager in the relative topology of P. It is known (see e.g. [8]) that in ZFC there always exists a perfectly meager set of size $\aleph_1$, and that it is consistent with ZFC that $\aleph_1 < \mathfrak{c}$ and every perfectly meager set has cardinality less or equal $\aleph_1$.

1. A-permitted sets

We will need some notations. Let $\|x\|$ denote the distance of a real x to the nearest integer, i.e. $\|x\| = \min\{|x-k| : k \in \mathbb{Z}\}$. Clearly the sequence $\{\sin \pi n_k x\}_{k\in\mathbb{N}}$ converges to 0 if and only if the sequence $\{\|n_k x\|\}_{k\in\mathbb{N}}$ does. Also $\|-x\| = \|x\|$ and $\|x\| - \|y\| \le \|x+y\| \le \|x\| + \|y\|$, for all $x, y \in \mathbb{R}$.

For $a \in \mathbb{N}^{\mathbb{N}}$, denote

$$A(a) = \left\{x : \lim_{n\to\infty} \|a(n)x\| = 0\right\}.$$

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Put $Seq = \{a \in \mathbb{N}^{\mathbb{N}} : a \text{ is increasing} \wedge a(0) = 1\}$, and

$$S = \left\{ a \in Seq : \lim_{n\to\infty} \frac{a(n)}{a(n+1)} = 0 \right\}.$$

It is easy to see that the family $\{A(a) : a \in S\}$ is a base of $\mathcal{A}$, i.e. for every $X \in \mathcal{A}$ there exists $a \in S$ such that $X \subseteq A(a)$. It was proved in [4] that if $a \in S$ then the set $A(a)$ intersects any nonempty open set in a set of the cardinality $\mathfrak{c}$.

The following notion was introduced in [5]. For $a \in Seq$, $m \in \mathbb{Z}$, and $z \in \mathbb{Z}^{\mathbb{N}}$ we say that z is a *good expansion of m by a*, if

$$m = \sum_{n\in\mathbb{N}} z(n)a(n) \text{ and for all } n \in \mathbb{N}, \quad \left|\sum\nolimits_{j<n} z(j)a(j)\right| \le \frac{a(n)}{2}.$$

Clearly if $z \in \mathbb{Z}^{\mathbb{N}}$ is a good expansion then the set $\{n \in \mathbb{N} : z(n) \neq 0\}$ is finite. The following two theorems were proved in [5]. We repeat the proofs of both theorems in the next section of this paper, now with more details and some corrections made.

Theorem 1.1. *For all $a \in Seq$ and every $m \in \mathbb{Z}$, there exists a good expansion of m by a.*

Theorem 1.2. *Let $a, b \in S$. For all $k \in \mathbb{N}$, let $z_k \in \mathbb{Z}^{\mathbb{N}}$ be a good expansion of $b(k)$ by a. Then $A(a) \subseteq A(b)$ if and only if*

(1) $\forall n \in \mathbb{N}\ \forall^\infty k \in \mathbb{N}\ z_k(n) = 0$, *and*
(2) $\exists m \in \mathbb{N}\ \forall k \in \mathbb{N}\ \sum_{n\in\mathbb{N}} |z_k(n)| \le m$.

Let us note that a special case of Theorem 1.2 for $a(n) = 2^{2^n}$ was proved by J. Arbault in [1].

For $a, b \in S$, let $a \prec b$ if and only if the conditions (1) and (2) from the previous theorem hold true. We obtain the following characterization of A-permitted sets.

Theorem 1.3. *A set $X \subseteq \mathbb{R}$ is permitted for the family $\mathcal{A}$ if and only if for every $a \in S$ there exists $b \in S$ such that $a \prec b$ and $X \subseteq A(b)$.*

Proof. If $a \in S$ and X is A-permitted then $X \cup A(a) \in \mathcal{A}$, and hence $X \cup A(a) \subseteq A(b)$ for some $b \in S$. It follows that $X \subseteq A(b)$ and $a \prec b$. $\square$

The next result of harmonic analysis is known as Kronecker's theorem. For a proof see e.g. [3] or [7].

Theorem 1.4. *Let $x_1, \dots, x_n \in \mathbb{R}$ be linearly independent over $\mathbb{Q}$, i.e. if $q_1x_1 + \dots + q_nx_n = 0$ for some rational $q_1, \dots, q_n$, then $q_1 = \dots = q_n = 0$. Let $y_1, \dots, y_n \in \mathbb{R}$ and $\varepsilon > 0$. Then there exist infinitely many natural numbers m such that for all i, $\|mx_i - y_i\| < \varepsilon$.*

Our main tool is the following lemma.

Lemma 1.5. *For every perfect set $P \subseteq \mathbb{R}$ there exists $a \in S$ such that for all $b \in S$, if $a \prec b$ then the set*

$$Q = \left\{ x \in P : \limsup_{k\to\infty} \|b(k)x\| = \frac{1}{2} \right\}$$

is dense in P.

Proof. Let P be a perfect set. We use a fact that any open set containing an element of P contains uncountably many elements of P.

Let $\{H_n\}_{n\in\mathbb{N}}$ be a sequence of open intervals such that $\lim_{n\to\infty} \operatorname{diam}(H_n) = 0$, $P \subseteq \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} H_n$, and for all $n \in \mathbb{N}$, $P \cap H_n \neq \emptyset$. Such sequence can be selected from a base of topology on $\mathbb{R}$ consisting from open intervals. Fix a non-increasing sequence of positive reals $\{\varepsilon_n\}_{n\in\mathbb{N}}$ converging to 0. By an induction on n, we will define $a \in S$ and a sequence $\{\mathcal{I}_n\}_{n\in\mathbb{N}}$ of finite families of open intervals such that for all n,

(i) $H_n \in \mathcal{I}_n$,
(ii) for all $I \in \mathcal{I}_n$, $P \cap I \neq \emptyset$,
(iii) for every $I \in \mathcal{I}_n$ there exists $J \in \mathcal{I}_{n+1}$ such that $J \subseteq I$ and for all $x \in J$, $\|a(n+1)x\| \leq \dfrac{\varepsilon_n}{n+1}$,
(iv) for every $I \in \mathcal{I}_n$ and for every $i \in \{1,\dots,n+1\}$ there exists $J \in \mathcal{I}_{n+1}$ such that $J \subseteq I$ and for all $x \in J$, $\|ia(n+1)x\| \geq \dfrac{1}{2} - \dfrac{\varepsilon_n}{n+1}$.

Put $a(0) = 1$ and $\mathcal{I}_0 = \{H_0\}$. If $a(n)$ and $\mathcal{I}_n$ are already defined, then for every $I \in \mathcal{I}_n$ and every $i \in \{0,\dots,n+1\}$ let us select a point $x_i^I \in P \cap I$ in such a way that the set

$$\left\{x_i^I : I \in \mathcal{I}_n \wedge i \in \{0,\dots,n+1\}\right\}$$

will be linearly independent on $\mathbb{Q}$. This is possible since the vector space over $\mathbb{Q}$ generated by a finite set is countable, while $P \cap I$ is uncountable for every $I \in \mathcal{I}_n$. By Theorem 1.4, there exists $a(n+1)$ such that $a(n)/a(n+1) \leq 1/(n+1)$ and for all $I \in \mathcal{I}_n$ and $i \in \{1,\dots,n+1\}$,

$$\left\|a(n+1)x_0^I\right\| \leq \frac{\varepsilon_n}{2(n+1)} \quad \text{and} \quad \left\|ia(n+1)x_i^I - \frac{1}{2}\right\| \leq \frac{\varepsilon_n}{2(n+1)}.$$

Let $\delta > 0$ be such that $a(n+1)\delta \leq \varepsilon_n/2(n+1)^2$ and for all $I \in \mathcal{I}_n$, $\delta < \operatorname{diam}(I)$. For every $I \in \mathcal{I}_n$ and $i \in \{0,\dots,n+1\}$, let $J_i^I \subseteq I$ be an open interval such that $\operatorname{diam}(J_i^I) = \delta$ and $x_i^I \in J_i^I$. Then for all $x \in J_0^I$ we have

$$\|a(n+1)x\| \leq \left\|a(n+1)x_0^I\right\| + a(n+1)\delta \leq \frac{\varepsilon_n}{2(n+1)} + \frac{\varepsilon_n}{2(n+1)^2} \leq \frac{\varepsilon_n}{n+1},$$

and for all $i \in \{1,\dots,n+1\}$ and $x \in J_i^I$,

$$\|ia(n+1)x\| \geq \left\|ia(n+1)x_i^I\right\| - ia(n+1)\delta \geq \frac{1}{2} - \frac{\varepsilon_n}{2(n+1)} - \frac{\varepsilon_n}{2(n+1)} = \frac{1}{2} - \frac{\varepsilon_n}{n+1}.$$

Put $\mathcal{I}_{n+1} = \{H_{n+1}\} \cup \left\{J_i^I : I \in \mathcal{I}_n \wedge i \in \{0,\dots,n+1\}\right\}$. We proceed this way for all $n \in \mathbb{N}$. We clearly obtain $a \in S$.

Let $b \in S$ be such that $a \prec b$. For all $k \in \mathbb{N}$, let $z_k \in \mathbb{Z}^{\mathbb{N}}$ be a good expansion of $b(k)$ by a. By Theorem 1.2, there exists $m \in \mathbb{N}$ such that for all k, $\sum_{n\in\mathbb{N}} |z_k(n)| \leq m$. We denote $\operatorname{supp} z_k = \{n \in \mathbb{N} : z_k(n) \neq 0\}$.

To prove that Q is dense in P, let us take an open interval I such that $P \cap I \neq \emptyset$. We need to show that $Q \cap I \neq \emptyset$.

Let us define sequences $\{n_j\}_{j\in\mathbb{N}}$, $\{k_j\}_{j\in\mathbb{N}}$, and $\{J_n\}_{n\geq n_0}$ as follows. Find n_0 such that $n_0 \geq m$ and $\operatorname{cl}(H_{n_0}) \subseteq I$. Put $J_{n_0} = H_{n_0}$.

Having defined n_j and $J_{n_j} \in \mathcal{I}_{n_j}$, find k_j such that $\min \operatorname{supp} z_{k_j} > n_j$ and put $n_{j+1} = \max \operatorname{supp} z_{k_j}$. For all n such that $n_j < n < n_{j+1}$, let us find $J_n \in \mathcal{I}_n$ such that $J_n \subseteq J_{n-1}$ and for all $x \in J_n$,

$$\|a(n)x\| \leq \frac{\varepsilon_n}{n+1} \leq \frac{\varepsilon_{n_j}}{m}$$

Finally, for $n = n_{j+1}$ let us find $J_n \in \mathcal{I}_n$ such that $J_n \subseteq J_{n-1}$ and for all $x \in J_n$,

$$\left\|z_{k_j}(n)a(n)x\right\| \geq \frac{1}{2} - \frac{\varepsilon_n}{n+1} \geq \frac{1}{2} - \frac{\varepsilon_{n_j}}{m}.$$

Since

$$b(k_j) = \sum_{n_j < n \leq n_{j+1}} z_{k_j}(n)a(n) \quad \text{and} \quad \sum_{n_j < n < n_{j+1}} \left|z_{k_j}(n)\right| \leq m - 1,$$

we obtain that for all $x \in J_{n_{j+1}}$, and thus also for all $x \in \mathrm{cl}(J_{n_{j+1}})$,

$$\|b(k_j)x\| \geq \left\|z_{k_j}(n_{j+1})a(n_{j+1})x\right\| - \sum_{n_j < n < n_{j+1}} \left|z_{k_j}(n)\right| \|a(n)x\| \geq \frac{1}{2} - \varepsilon_{n_j}.$$

We can see that if $x \in \bigcap_{n \geq n_0} \mathrm{cl}(J_n)$ then $x \in Q \cap I$. □

We are now ready to prove the main result of this paper.

Theorem 1.6. *Let X be permitted for the family $\mathcal{A}$. Then X is perfectly meager.*

Proof. Let P be a perfect set. We will show that X is meager in P, i.e. that X is a countable union of sets which are nowhere dense in P.

By Lemma 1.5 there exists $a \in S$ such that if $b \in S$ and $a \prec b$ then the set $Q = \left\{x \in P : \limsup_{k\to\infty} \|b(k)x\| = \frac{1}{2}\right\}$ is dense in P. Since X is A-permitted, by Theorem 1.3 there exists such $b \in S$ satisfying $a \prec b$ and $X \subseteq A(b)$.

For $n \in \mathbb{N}$, denote $X_n = \{x \in \mathbb{R} : \forall k \geq n \;\; \|b(k)x\| \leq 1/4\}$. Clearly every X_n is closed and $X \subseteq \bigcup_{n\in\mathbb{N}} X_n$. It suffices to show that X_n is nowhere dense in P, for every n.

Let $n \in \mathbb{N}$ and let G be an open set such that $G \cap P \neq \emptyset$. Then there exists $x \in Q \cap G$. We have $x \notin X_n$ and since X_n is closed, there exists an open set $H \subseteq G$ such that $x \in H$ and $H \cap X_n = \emptyset$. Hence X_n is nowhere dense in P. □

2. Proof of Theorems 1.1 and 1.2

Theorem 1.1 was proved in [5] as Lemma 2. We repeat the proof here for the completeness.

Proof of Theorem 1.1. Let $a \in S$ and $m \in \mathbb{Z}$. We find a good expansion $z \in \mathbb{Z}^{\mathbb{N}}$ of m by a as follows. Find some $k \in \mathbb{N}$ such that $|m| \leq a(k)/2$ and put $z(n) = 0$ for all $n > k$. For $n \leq k$, $z(n)$ will be defined by induction on n going from k to 0.

Put $m_{k+1} = m$. For $n \leq k$, let $z(n)$ be the nearest integer to $m_{n+1}/a(n)$. Put $m_n = m_{n+1} - z(n)a(n)$. Since $a(0) = 1$, we obtain $m_0 = 0$, and thus for all $n \leq k$, $m_{n+1} = \sum_{j\leq n} z(j)a(j)$. It follows that z is an expansion of m by a.

For $n \leq k$ we have $|m_{n+1}/a(n) - z(n)| \leq 1/2$, and hence

$$\left|\sum_{j<n} z(j)a(j)\right| = |m_{n+1} - z(n)a(n)| \leq \frac{a(n)}{2}.$$

Since a is increasing, for $n > k$ we obtain

$$\left|\sum_{j<n} z(j)a(j)\right| = |m| \leq \frac{a(k)}{2} < \frac{a(n)}{2}.$$

Hence z is a good expansion. $\square$

Here we provide a corrected version of the proof of Theorem 1.2, proved in [5] as Theorem 4. The proof is now rearranged, one part of the proof was completely rewritten. One direction of the proof is easy.

Lemma 2.1. *Let $a \in S$, $b \in Seq$, and for all $k \in \mathbb{Z}$, let $z_k \in \mathbb{Z}^{\mathbb{N}}$ be a good expansion of $b(k)$ by a. Assume that*
(1) $\forall n \in \mathbb{N}\ \forall^\infty k \in \mathbb{N}\ z_k(n) = 0$, *and*
(2) $\exists m \in \mathbb{N}\ \forall k \in \mathbb{N}\ \sum_{n\in\mathbb{N}} |z_k(n)| \leq m$.
Then $A(a) \subseteq A(b)$.

Proof. Let $m > 0$ be such that for all k, $\sum_{n\in\mathbb{N}} |z_k(n)| \leq m$. If $x \in A(a)$ and if $\varepsilon > 0$ is given, then there exists n_0 such that for all $n \geq n_0$, $\|a(n)x\| \leq \varepsilon/m$. There exists k_0 such that for all $n < n_0$ and $k \geq k_0$, $z_k(n) = 0$. For all $k \geq k_0$ we have

$$\|b(k)x\| \leq \sum_{n\in\mathbb{N}} |z_k(n)|\,\|a(n)x\| \leq \frac{\varepsilon}{m}\sum_{n\in\mathbb{N}} |z_k(n)| \leq \varepsilon,$$

hence $x \in A(b)$. $\square$

The proof of the other direction splits into several lemmas.

Lemma 2.2. *If z is a good expansion by a, then*

$$|z(n)| \leq \frac{1}{2}\left(1 + \frac{a(n+1)}{a(n)}\right)$$

for all $n \in \mathbb{N}$.

Proof. For $n \in \mathbb{N}$, we have

$$|z(n)a(n)| \leq \left|\sum_{j<n} z(j)a(j)\right| + \left|\sum_{j\leq n} z(j)a(j)\right| \leq \frac{a(n)}{2} + \frac{a(n+1)}{2},$$

hence

$$|z(n)| \leq a(n)^{-1}\left(\frac{a(n)}{2} + \frac{a(n+1)}{2}\right) = \frac{1}{2}\left(1 + \frac{a(n+1)}{a(n)}\right).$$

$\square$

By an interval we will mean a closed and bounded interval on the real line having a nonempty interior.

Lemma 2.3. *Let $a \in Seq$, $n \in \mathbb{N}$, and let $a(n)/a(n+1) \leq 1/4$. Then for every interval I such that $\operatorname{diam}(I) = 4/(3a(n))$ there exists an interval $J \subseteq I$ such that $\operatorname{diam}(J) = 4/(3a(n+1))$ and for all $x \in J$,*

$$\|a(n)x\| \leq \frac{2a(n)}{3a(n+1)}.$$

Proof. Take an interval I' of the length $1/a(n)$ having the same center as the interval I, and find $x_0 \in I'$ such that $\|a(n)x_0\| = 0$. Let J be an interval of the length $4/(3a(n+1))$ with the center x_0. For all $x \in J$ we obtain

$$|x - x_0| \leq \frac{2}{3a(n+1)} \leq \frac{1}{6a(n)} = \frac{\operatorname{diam}(I) - \operatorname{diam}(I')}{2},$$

hence $x \in I$ and $\|a(n)x\| \leq a(n)\,|x - x_0| \leq \dfrac{2a(n)}{3a(n+1)}$. $\square$

Lemma 2.4. *Let $a \in Seq$, and let $n \in \mathbb{N}$ be such that $a(n)/a(n+1) \le 1/4$. Let $z \in \mathbb{Z}^{\mathbb{N}}$ be a good expansion by a such that $|z(n)| \ge 2$. Then for every interval I such that* $\operatorname{diam}(I) = 4/(3a(n))$ *there exists an interval $J \subseteq I$ such that* $\operatorname{diam}(J) = 4/(3a(n+1))$ *and for all $x \in J$,*

$$\|a(n)x\| \le \frac{2a(n)}{3a(n+1)} + \frac{1}{|z(n)| - \frac{1}{2}} \quad \text{and} \quad \left\| \sum_{j\le n} z(j)a(j)x \right\| \ge \frac{1}{6}.$$

Proof. Take an interval I' of the length $1/a(n)$ having the same center as the interval I, and find $x_0 \in I'$ such that $\|a(n)x_0\| = 0$. Put $m = \left|\sum_{j\le n} z(j)a(j)\right|$. Since $z(n) \ge 2$, we have

$$m \ge |z(n)|\, a(n) - \left|\sum_{j<n} z(j)a(j)\right| \ge \left(|z(n)| - \tfrac{1}{2}\right) a(n) \ge \frac{3a(n)}{2}.$$

We have $1/m < \operatorname{diam}(I')$, hence there exists $x_1 \in I'$ such that $\|mx_1\| = 1/2$ and $|x_1 - x_0| \le 1/m$. Let J be an interval of the length $4/(3a(n+1))$ with the center x_1. For all $x \in J$ we have

$$|x - x_1| \le \frac{2}{3a(n+1)} \le \frac{1}{6a(n)} = \frac{\operatorname{diam}(I) - \operatorname{diam}(I')}{2},$$

hence $J \subseteq I$. By the definition of a good expansion we have $m \le a(n+1)/2$, thus if $x \in J$ then $\|mx\| \ge \|mx_0\| - m\,|x - x_1| \ge 1/2 - 1/3 = 1/6$. We obtain that for all $x \in J$, $|x - x_0| \le |x - x_1| + |x_1 - x_0| \le 2a(n)/(3a(n+1)) + 1/m$, hence

$$\|a(n)x\| \le a(n)\,|x - x_0| \le \frac{2a(n)}{3a(n+1)} + \frac{1}{|z(n)| - \frac{1}{2}}.$$

□

Lemma 2.5. *Let $a \in Seq$, and let $n \in \mathbb{N}$ be such that $a(n)/a(n+1) \le 1/8$. Let $z \in \mathbb{Z}^{\mathbb{N}}$ be a good expansion by a. If I is an interval such that* $\operatorname{diam}(I) = 4/(3a(n))$ *and for all $x \in I$, $\left\|\sum_{j<n} z(j)a(j)x\right\| \ge 1/6$, then there exists an interval $J \subseteq I$ such that* $\operatorname{diam}(J) = 4/(3a(n+1))$ *and for all $x \in J$,*

$$\|a(n)x\| \le \frac{4a(n)}{3a(n+1)} \quad \text{and} \quad \left\|\sum_{j\le n} z(j)a(j)x\right\| \ge \frac{1}{6}.$$

Proof. Take an interval I' of the length $1/a(n)$ having the same center as the interval I, and find $x_0 \in I'$ such that $\|a(n)x_0\| = 0$. Put $m = \left|\sum_{j\le n} z(j)a(j)\right|$. Since $\left\|\sum_{j<n} z(j)a(j)x_0\right\| \ge 1/6$ and $\|a(n)x_0\| = 0$, we have $\|mx_0\| \ge 1/6$. Denote J' the longest interval containing x_0 on which the condition $\|mx\| \ge 1/6$ is satisfied. By the definition of a good expansion, we have $m \le a(n+1)/2$, hence $\operatorname{diam}(J') = 2/(3m) \ge 4/(3a(n+1))$. Let J be an interval of the length $4/(3a(n+1))$ such that $J \subseteq J'$ and $x_0 \in J$. For all $x \in J$ we obtain

$$|x - x_0| \le \frac{4}{3a(n+1)} \le \frac{1}{6a(n)} = \frac{\operatorname{diam}(I) - \operatorname{diam}(I')}{2},$$

hence $x \in I$ and $\|a(n)x\| \le a(n)\,|x - x_0| \le \dfrac{4a(n)}{3a(n+1)}$. □

Lemma 2.6. *Let $a \in S$, $b \in Seq$, and for all $k \in \mathbb{N}$, let $z_k \in \mathbb{Z}^{\mathbb{N}}$ be a good expansion of $b(k)$ by a. Assume that the set $\{|z_k(n)| : k, n \in \mathbb{N}\}$ is unbounded. Then there exists $x \in A(a)$ such that $x \notin A(b)$.*

Proof. We will find increasing sequences of natural numbers $\{n_i\}_{i\in\mathbb{N}}$, $\{k_i\}_{i\in\mathbb{N}}$ such that

(i) for all $n \geq n_0$, $a(n)/a(n+1) \leq 1/8$,
(ii) for all $i \in \mathbb{N}$, $|z_{k_i}(n_i)| \geq 2$,
(iii) $\lim_{i\to\infty} |z_{k_i}(n_i)| = \infty$,
(iv) for all $i \in \mathbb{N}$ and for all $n \geq n_{i+1}$, $z_{k_i}(n) = 0$.

The sequences $\{n_i\}_{i\in\mathbb{N}}$, $\{k_i\}_{i\in\mathbb{N}}$ can be defined by induction as folows. Let $m_0 \in \mathbb{N}$ be such that for all $n \geq m_0$, $a(n)/a(n+1) \leq 1/8$. Since by Lemma 2.2, the set $\{|z_k(n)| : k \in \mathbb{N}\}$ is bounded for every $n \in \mathbb{N}$, there exist n_0 and k_0 such that $n_0 \geq m_0$ and $|z_{k_0}(n_0)| \geq 2$. If n_i and k_i are already defined, find m_{i+1} such that for all $n \geq m_{i+1}$, $z_{k_i}(n) = 0$. Again, there exist n_{i+1} and k_{i+1} such that $n_{i+1} \geq m_{i+1}$, $k_{i+1} > k_i$, and $\left|z_{k_{i+1}}(n_{i+1})\right| > |z_{k_i}(n_i)|$. It can be easily checked that the sequences $\{n_i\}_{i\in\mathbb{N}}$ and $\{k_i\}_{i\in\mathbb{N}}$ are increasing and the conditions (i)–(iv) are satisfied.

Let us now define a sequence of intervals $\{I_n\}_{n\geq n_0}$. We start with an arbitrary interval I_{n_0} such that $\operatorname{diam}(I_{n_0}) = 4/(3a(n_0))$.

For $n \geq n_0$, let I_n be an interval of the length $4/(3a(n))$. If $n = n_i$ for some i then by Lemma 2.4 there exists an interval $I_{n+1} \subseteq I_n$ of the length $4/(3a(n+1))$ such that for all $x \in I_{n+1}$,

$$\|a(n)x\| \leq \frac{2a(n)}{3a(n+1)} + \frac{1}{|z_{k_i}(n_i)| - \frac{1}{2}} \text{ and } \left\|\textstyle\sum_{j\leq n} z_{k_i}(j)a(j)x\right\| \geq \frac{1}{6}.$$

Otherwise, $n_i < n < n_{i+1}$ for some i, and we have $\left\|\sum_{j<n} z_{k_i}(j)a(j)x\right\| \geq 1/6$ for all $x \in I_n$. By Lemma 2.5 there exists an interval $I_{n+1} \subseteq I_n$ of the length $4/(3a(n+1))$ such that for all $x \in I_{n+1}$,

$$\|a(n)x\| \leq \frac{4a(n)}{3a(n+1)} \text{ and } \left\|\textstyle\sum_{j\leq n} z_{k_i}(j)a(j)x\right\| \geq \frac{1}{6}.$$

Let $x \in \bigcap_{n\geq n_0} I_n$. From $a \in S$ and the condition (iii) we obtain $x \in A(a)$. For all i, the condition (iv) implies that

$$\|b(k_i)x\| = \left\|\textstyle\sum_{j<n_{i+1}} z_{k_i}(j)a(j)x\right\| \geq \frac{1}{6},$$

hence $x \notin A(b)$. □

Lemma 2.7. *Let $a \in Seq$, and let $n \in \mathbb{N}$ be such that $a(n)/a(n+1) \leq 1/16$. Let $z \in \mathbb{Z}^{\mathbb{N}}$ be a good expansion by a such that $z(n) \neq 0$. Let c, ε be non-negative reals, and let $\varepsilon \leq 1/24$. If I is an interval such that* $\operatorname{diam}(I) = 4/(3a(n))$ *and for all $x \in I$, $\left\|\sum_{j<n} z(j)a(j)x\right\| \geq c$, then there exists an interval $J \subseteq I$ such that* $\operatorname{diam}(J) = 4/(3a(n+1))$ *and for all $x \in J$,*

$$\|a(n)x\| \leq \frac{4a(n)}{3a(n+1)} + 2\varepsilon \textit{ and } \left\|\textstyle\sum_{j\leq n} z(j)a(j)\right\| \geq \min\left\{\frac{1}{6}, c+\varepsilon\right\}.$$

Proof. Let I', x_0, and m be as in Lemma 2.5. Clearly $\|mx_0\| \geq c$. By the definition of a good expansion we have $m \leq a(n+1)/2$. Since $z(n) \neq 0$, we also have

$$m \geq a(n) - \left|\sum_{j<n} z(j)a(j)\right| \geq \frac{a(n)}{2}.$$

We will consider two cases.

(a) Assume that for some x_1, $\|mx_1\| \geq 1/6$ and $|x_1 - x_0| \leq \varepsilon/m$. Then there exists an interval J containing x_1 such that $\operatorname{diam}(J) = 4/(3a(n+1))$ and for all $x \in J$, $\|mx\| \geq 1/6$. For $x \in J$ we obtain

$$|x - x_0| \leq |x - x_1| + |x_1 - x_0| \leq \frac{4}{3a(n+1)} + \frac{\varepsilon}{m} \leq \frac{1}{6a(n)} = \frac{\operatorname{diam}(I) - \operatorname{diam}(I')}{2}.$$

Hence $J \subseteq I$.

(b) If $\|mx\| < 1/6$ for all x such that $|x - x_0| \leq \varepsilon/m$, then there exists x_1 such that $\|mx_1\| = \|mx_0\| + \varepsilon$ and $|x_1 - x_0| = \varepsilon/m$. Similarly as in the previous case, there exists an interval J containing x_1 such that $\operatorname{diam}(J) = 4/(3a(n+1))$ and for all $x \in J$, $\|mx\| \geq \|mx_1\|$. Again, we obtain $J \subseteq I$.

In both cases we have shown that for all $x \in J$, $\|mx\| \geq \min\{1/6, c + \varepsilon\}$ and

$$\|a(n)x\| \leq a(n)\,|x - x_0| \leq \frac{4a(n)}{3a(n+1)} + 2\varepsilon.$$

□

Lemma 2.8. *Let $a \in S$, $b \in Seq$, and for all $k \in \mathbb{N}$, let $z_k \in \mathbb{Z}^{\mathbb{N}}$ be a good expansion of $b(k)$ by a and $s_k = |\{n \in \mathbb{N} : z_k(n) \neq 0\}|$. Assume that the set $\{s_k : k \in \mathbb{N}\}$ is unbounded. Then there exists $x \in A(a)$ such that $x \notin A(b)$.*

Proof. We will find increasing sequences $\{n_i\}_{i\in\mathbb{N}}$, $\{k_i\}_{i\in\mathbb{N}}$ such that

(i) for all $n \geq n_0$, $a(n)/a(n+1) \leq 1/16$,
(ii) for all $i \in \mathbb{N}$, $s_{k_i} \geq n_i + i + 4$,
(iii) for all $i \in \mathbb{N}$ and for all $n \geq n_{i+1}$, $z_{k_i}(n) = 0$.

The sequences $\{n_i\}_{i\in\mathbb{N}}$, $\{k_i\}_{i\in\mathbb{N}}$ can be defined by induction as follows. Take n_0 such that for all $n \geq n_0$, $a(n)/a(n+1) \leq 1/16$, and find k_0 such that $s_{k_0} \geq n_0 + 4$. If n_i, k_i are already defined, then we can find $n_{i+1} > n_i$ such that for all $n \geq n_{i+1}$, $z_{k_i}(n) = 0$. By the assumption there exists $k_{i+1} > k_i$ such that $s_{k_{i+1}} \geq n_{i+1} + i + 5$.

For $i \in \mathbb{N}$, let us denote $m_i = |\{n : n \geq n_i \wedge z_{k_i}(n) \neq 0\}|$. From (ii) it follows that $m_i \geq i + 4$, hence $\lim_{i\to\infty} m_i = \infty$. Put $\varepsilon_i = 1/(6m_i)$. We have $m_i \geq 4$, hence $\varepsilon_i \leq 1/24$.

Let us define a sequence of intervals $\{I_n\}_{n\geq n_0}$. We start with an arbitrary interval I_{n_0} such that $\operatorname{diam}(I_{n_0}) = 4/(3a(n_0))$.

Let $n \geq n_0$ and let I_n be an interval of the length $4/(3a(n))$. Find i such that $n_i \leq n < n_{i+1}$ and put

$$c_n = \min\left\{\left\|\sum_{j<n} z_{k_i}(j)a(j)x\right\| : x \in I_n\right\}.$$

If $z_{k_i}(n) = 0$ then by Lemma 2.3 there exists an initerval $I_{n+1} \subseteq I_n$ of the length $4/(3a(n+1))$ such that for all $x \in I_{n+1}$, $\|a(n)x\| \leq 2a(n)/(3a(n+1))$. Clearly also $\left\|\sum_{j\leq n} z_{k_i}(j)a(j)x\right\| = \left\|\sum_{j<n} z_{k_i}(j)a(j)x\right\| \geq c_n$.

If $z_{k_i}(n) \neq 0$ then by Lemma 2.7 there exists an interval $I_{n+1} \subseteq I_n$ of the length $4/(3a(n+1))$ such that for all $x \in I_{n+1}$,

$$\|a(n)x\| \leq \frac{4a(n)}{3a(n+1)} + 2\varepsilon_i \text{ and } \left\|\sum_{j\leq n} z_{k_i}(j)a(j)x\right\| \geq \min\left\{\frac{1}{6}, c_n + \varepsilon_i\right\}.$$

Let $x \in \bigcap_{n\geq n_0} I_n$. Since $a \in S$ and $\lim_{i\to\infty} \varepsilon_i = 0$, we have $x \in A(a)$. For all $i \in \mathbb{N}$, the condition (iii) implies that

$$\|b(k_i)x\| = \left\|\sum_{j<n_{i+1}} z_{k_i}(j)a(j)x\right\| \geq \min\left\{\frac{1}{6}, m_i\varepsilon_i\right\} = \frac{1}{6},$$

since we have m_i-times increased the value $c_{n_i} \geq 0$ by ε_i. Hence $x \notin A(b)$. □

Lemma 2.9. *Let $a \in S$, $b \in Seq$, and for all $k \in \mathbb{N}$, let $z_k \in \mathbb{Z}^{\mathbb{N}}$ be a good expansion of $b(k)$ by a. Assume that there exist $t \in \mathbb{N}$ and an infinite set $K \subseteq \mathbb{N}$ such that for all $k \in K$, $z_k(t) \neq 0$, and for every $n > t$, the set $\{k \in K : z_k(n) \neq 0\}$ is finite. Then there exists $x \in A(a)$ such that $x \notin A(b)$.*

Proof. By Lemma 2.2, the set $\{z_k(n) : k \in \mathbb{N}\}$ is finite for every $n \in \mathbb{N}$, hence there exist integers $y(0), \ldots, y(t)$ and an infinite set $L \subseteq K$ such that for all $k \in L$ and $n \leq t$, $z_k(n) = y(n)$. Denote $m = \sum_{n\leq t} y(n)a(n)$. We will find increasing sequences $\{n_i\}_{i\in\mathbb{N}}$, $\{k_i\}_{i\in\mathbb{N}}$ such that

- (i) $n_0 > t$,
- (ii) for all $n \geq n_0$, $a(n)/a(n+1) \leq 1/8$,
- (iii) for all $i \in \mathbb{N}$, $k_i \in L$,
- (iv) for all $i \in \mathbb{N}$, $z_{k_i}(n_i) \neq 0$,
- (v) for all $i, n \in \mathbb{N}$, if $t < n < n_i$ or $n \geq n_{i+1}$ then $z_{k_i}(n) = 0$.

The sequences $\{n_i\}_{i\in\mathbb{N}}$, $\{k_i\}_{i\in\mathbb{N}}$ can be defined by induction as follows. Let $m_0 \geq t$ be such that for all $n > m_0$, $a(n)/a(n+1) \leq 1/8$. Find an infinite set $K_0 \subseteq L$ such that if $k \in K_0$ and $t < n \leq m_0$ then $z_k(n) = 0$. Let us take $k_0 \in K_0$ such that for some $n > m_0$, $z_{k_0}(n) \neq 0$. Put $n_0 = \min\{n > m_0 : z_{k_0}(n) \neq 0\}$.

If n_i, k_i, m_i, and K_i are already defined, let m_{i+1} be such that for all $n > m_{i+1}$, $z_{k_i}(n) = 0$. There exists an infinite set $K_{i+1} \subseteq K_i$ such that if $k \in K_{i+1}$ and $t < n \leq m_{i+1}$ then $z_k(n) = 0$. Let us take $k_{i+1} \in K_{i+1}$ such that $k_{i+1} > k_i$ and $z_{k_{i+1}}(n) \neq 0$ for some $n > m_{i+1}$. Put $n_{i+1} = \min\left\{n > m_{i+1} : z_{k_{i+1}}(n) \neq 0\right\}$.

Since for all i, $z_{k_i}(n_i) \neq 0$ and $z_{k_i}(n) = 0$ for all $n > m_{i+1}$, we have $n_i \leq m_{i+1}$, and thus $n_{i+1} > n_i$. We also have $k_{i+1} > k_i$ for all i, hence the sequences $\{n_i\}_{i\in\mathbb{N}}$ and $\{k_i\}_{i\in\mathbb{N}}$ are increasing. The condition $n_0 > m_0$ ensures that (i) and (ii) are satisfied; the conditions (iii)–(v) are ensured by the choice of K_{i+1} and n_{i+1}.

Take an interval I of the length $2/(3\,|m|)$ such that for all $x \in I$, $\|mx\| \geq 1/6$. We will define a sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$ by induction. Since $n_0 \geq t+1$ and $|m| \leq a(t+1)/2$, there exists an interval $I_0 \subseteq I$ such that $\operatorname{diam}(I_0) = 4/(3a(n_0))$. For $n \geq n_0$, let I_n be an interval of the length $4/(3a(n))$, and let i be such that $n_i \leq n < n_{i+1}$. If $n = n_i$, then for all $x \in I_n$ we have

$$\left\|\sum_{j<n} z_{k_i}(j)a(j)x\right\| = \|mx\| \geq \frac{1}{6}.$$

By Lemma 2.5, there exists an interval $I_{n+1} \subseteq I_n$ of the length $4/(3a(n+1))$ such that for all $x \in I_{n+1}$,

$$\|a(n)x\| \leq \frac{4a(n)}{3a(n+1)} \text{ and } \left\|\sum_{j\leq n} z_{k_i}(j)a(j)x\right\| \geq \frac{1}{6}.$$

We can find such interval I_{n+1} for every n, $n_i \le n < n_{i+1}$.

Let $x \in \bigcap_{n \ge n_0} I_n$. Since for all $n \ge n_0$, $\|a(n)x\| \le 4a(n)/(3a(n+1))$, we have $x \in A(a)$. We also obtain that for all i,

$$\|b(k_i)x\| = \left\|\sum_{j<n_{i+1}} z_{k_i}(j)a(j)x\right\| \ge \frac{1}{6},$$

hence $x \notin A(b)$. □

Lemma 2.10. *Let $a \in S$, $b \in Seq$, and for all $k \in \mathbb{N}$, let $z_k \in \mathbb{Z}^{\mathbb{N}}$ be a good expansion of $b(k)$ by a. Assume that there exists $t \in \mathbb{N}$ such that the set $\{k \in \mathbb{N} : z_k(t) \ne 0\}$ is infinite, and that for every $n \in \mathbb{N}$ and every infinite set $K \subseteq \{k \in \mathbb{N} : z_k(n) \ne 0\}$ there exists $n' > n$ such that $\{k \in K : z_k(n') \ne 0\}$ is infinite. Then there exists $x \in A(a)$ such that $x \notin A(b)$.*

Proof. We will find increasing sequences $\{n_i\}_{i\in\mathbb{N}}$, $\{k_i\}_{i\in\mathbb{N}}$ such that

(i) for all $n \ge n_0$, $a(n)/a(n+1) \le 1/16$,
(ii) for all i, j such that $i \le j$, and for all $n \le n_i$, $z_{k_i}(n) = z_{k_j}(n)$,
(iii) for all i and for all n such that $n_0 \le n \le n_i$, $z_{k_i}(n) \ne 0$ if and only if $n = n_j$ for some $j \le i$.

The sequences $\{n_i\}_{i\in\mathbb{N}}$, $\{k_i\}_{i\in\mathbb{N}}$ can be defined by induction as follows. By the assumption, there exists $t \in \mathbb{N}$ such that for all $n \ge t$, $a(n)/a(n+1) \le 1/16$, and the set $K = \{k \in \mathbb{N} : z_k(t) \ne 0\}$ is infinite. Similarly as in Lemma 2.9, there exist integers $y(0), \dots, y(t)$ and an infinite set $L \subseteq K$ such that for all $n \le t$ and $k \in L$, $z_k(n) = y(n)$.

Put $n_0 = t$ and $K_0 = L$. Let us take arbitrary $k_0 \in K_0$.

If we have n_i, k_i, and K_i defined, let $n_{i+1} = \min\{n > n_i : \exists^\infty k \in K_i\ z_k(n) \ne 0\}$. We can find an integer $y(n_{i+1}) \ne 0$ and an infinite set $K_{i+1} \subseteq K_i$ such that for all $k \in K_{i+1}$, $z_k(n_{i+1}) = y(n_{i+1})$, and for all n such that $n_i < n < n_{i+1}$, $z_k(n) = 0$. Let us take $k_{i+1} \in K_{i+1}$ such that $k_{i+1} > k_i$.

Clearly the sequences $\{n_i\}_{i\in\mathbb{N}}$ and $\{k_i\}_{i\in\mathbb{N}}$ are increasing. Condition (i) is ensured by the choice of n_0, conditions (ii) and (iii) by the choice of n_{i+1} and K_{i+1}.

For $k \in \mathbb{N}$, let us denote $s_k = |\{n \in \mathbb{N} : z_k(n) \ne 0\}|$. The condition (iii) implies that for all i, $s_{k_i} \ge i + 1$, hence the set $\{s_k : k \in \mathbb{N}\}$ is unbounded. From Lemma 2.8 it follows that there exists $x \in A(a)$ such that $x \notin A(b)$. □

It remains to prove the second direction of Theorem 1.2.

Lemma 2.11. *Let $a \in S$, $b \in Seq$, and for all $k \in \mathbb{N}$, let $z_k \in \mathbb{Z}^{\mathbb{N}}$ be a good expansion of $b(k)$ by a. If $A(a) \subseteq A(b)$ then*

(1) $\forall n \in \mathbb{N}\ \forall^\infty k \in \mathbb{N}\ z_k(n) = 0$, *and*
(2) $\exists m \in \mathbb{N}\ \forall k \in \mathbb{N}\ \sum_{n\in\mathbb{N}} |z_k(n)| \le m$.

Proof. Assume that the condition (1) fails. Then there exists $t \in \mathbb{N}$ such that the set $K = \{n \in \mathbb{N} : z_k(t) \ne 0\}$ is infinite. In the case that there exist such t and K with an additional property that for all $n > t$, the set $k \in K : z_k(n) \ne 0\}$ is finite, Lemma 2.9 says that $A(a) \nsubseteq A(b)$. On the other side, if there is no such t and K then for every n and every infinite set $K \subseteq \{k : z_k(n) \ne 0\}$ there exists $n' > n$ such that the set $\{k \in K : z_k(n') \ne 0\}$ is infinite. Thus the assumptions of Lemma 2.10 are satisfied and hence $A(a) \nsubseteq A(b)$. We have proved that if $A(a) \subseteq A(b)$ then the condition (1) holds true.

Assume now that the condition (2) fails. There are two possibilities. Either the set $\{|z_k(n)| : n, k \in \mathbb{N}\}$ is unbounded, or the set $\{s_k : k \in \mathbb{N}\}$ is unbounded, where $s_k = |\{n \in \mathbb{N} : z_k(n) \neq 0\}|$. In the first case, Lemma 2.6 implies that $A(a) \nsubseteq A(b)$, in the latter the same follows from Lemma 2.8. Thus if $A(a) \subseteq A(b)$ then the condition (2) must hold true too. □

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