On microperiodic multifunctions

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Definition 1

Let $(X, \cdot)$ be a topological group. A function $f : X \to Y$ is called microperiodic if there exists a dense subset P of X such that

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Let $(X,\cdot)$ be a topological group, P dense subset of X. If $f : X \to \overline{\mathbb{R}}$ satisfying

$$f(px) \leqslant f(x), \qquad x \in X, p \in P, \tag{m}$$

is continuous at some $x_0 \in X$, then it is constant.

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Let $(X,\cdot)$ be a topological group. We say that a multifunction $F : X \to 2^Y$ is microperiodic if there exists a dense subset P of X such that

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Let $A, B \subset Y$. Multifunction $F : \mathbb{R} \to 2^Y$ of the form

$$F(x) = \begin{cases} A & \text{for } x \in \mathbb{Q}, \\ B & \text{for } x \notin \mathbb{Q} \end{cases}$$

is microperiodic.

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If (Y, d) is a metric space, $A, B \subset Y$, define

$$e(A, B) := \sup\{d(a, B) : \ a \in A\},$$
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1. $F_1(x) = \begin{cases} [0,1] & \textit{for } x \in \mathbb{Q}, \\ (0,1) & \textit{for } x \notin \mathbb{Q} \end{cases}$
2. $F_2(x) = \begin{cases} (0,1) & \textit{for } x \in \mathbb{Q}, \\ (0,1) \setminus \{\frac{1}{2}\} & \textit{for } x \notin \mathbb{Q} \end{cases}$
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Proof.

1. Define $f(x) := e(F(x_0), F(x))$, $x \in X$

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4. f is constant, therefore $e(F(x_0), F(x)) = f(x) = f(x_0) = e(F(x_0), F(x_0)) = 0$ for $x \in X$

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6. $g(x) := e(F(x), F(x_0))$, $x \in X$
7. $F(x) \subset \overline{F}(x_0) \subset \overline{F}(x)$

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Theorem 2

Assume that X is a topological group, Y a normed space, $F : X \to P_{fc}(Y)$ is a microperiodic multifunction. If F is lsc at x_0 and usc at x_0 (Y endowed with the weak topology), then it is constant.

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Assume that X is a topological group, Y a normed space, $F : X \to P_{fc}(Y)$ is a microperiodic multifunction. If F is lsc at x_0 and usc at x_0 (Y endowed with the weak topology), then it is constant.

Proof.

1. $y^* \in Y^*$

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1. $y^* \in Y^*$
2. $f_{y^*}(x) := \sigma(y^*, F(x)) = \sup\{(y^*, y) : \ y \in F(x)\}, x \in X$

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2. $f_{y^*}(x) := \sigma(y^*, F(x)) = \sup\{(y^*, y) : \ y \in F(x)\},\ x \in X$
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4. $f_{y^*}(px) \leqslant f_{y^*}(x), \, x \in X, \, p \in P$

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5. f_{y^*} is constant

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4. $f_{y^*}(px) \leqslant f_{y^*}(x)$, $x \in X$, $p \in P$
5. f_{y^*} is constant
6. $\sigma(y^*, F(x)) = \sigma(y^*, F(x_0))$ for $x \in X$

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4. $f_{y^*}(px) \leqslant f_{y^*}(x)$, $x \in X$, $p \in P$
5. f_{y^*} is constant
6. $\sigma(y^*, F(x)) = \sigma(y^*, F(x_0))$ for $x \in X$
7. by the separation theorem $F(x) = F(x_0)$, $x \in X$

□

Theorem 3

Assume that X is a topological group, Y a topological space, $F : X \to P(Y)$ a microperiodic multifunction with open values. If F is usc on some open neighborhood of x_0, then it is constant.

Let $(X, \mathcal{M})$ be a measurable space, (Y, d) a separable metric space, $F : X \longrightarrow 2^Y$

- F is weakly measurable (measurable for short) if the lower inverse image $F^-(U) := \{x \in X : \ F(x) \cap C \neq \emptyset\}$ is measurable for every open set $U \subset Y$
- F is strongly measurable if the lower inverse image $F^-(C) := \{x \in X : \ F(x) \cap C \neq \emptyset\}$ is measurable for every closed set $C \subset Y$

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Example 2

Let $A, B \subset Y$, $P \subset X$ is dense and for every $p \in P$ the following condition holds

$$px \in P, \ \text{for } x \in P \qquad \text{and} \qquad px \in X \setminus P, \ \text{for } x \in X \setminus P.$$

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- *F is microperiodic*
- *F strongly measurable iff $P \in \mathcal{M}$*

Let $(X,\mathcal{M})$ be a measurable space, (Y,d) a separable metric space, $F : X \longrightarrow 2^Y$

- F is weakly measurable (measurable for short) if the lower inverse image $F^-(U) := \{x \in X : \ F(x) \cap C \neq \emptyset\}$ is measurable for every open set $U \subset Y$
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Example 2

Let $A, B \subset Y$, $P \subset X$ is dense and for every $p \in P$ the following condition holds

$$px \in P, \ \text{for } x \in P \qquad \text{and} \qquad px \in X \setminus P, \ \text{for } x \in X \setminus P.$$

$$F(x) = \begin{cases} A & \text{for } x \in P, \\ B & \text{for } x \in X \setminus P \end{cases}.$$

- *F is microperiodic*
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Example 3

Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function with small graph (i.e. $\operatorname{card} f(\mathbb{R}) = \aleph_0$).

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$$F(x) = |f(x)| \cdot C = \{|f(x)| \cdot c : c \in C\}, \qquad x \in \mathbb{R}$$

Theorem 4

Let X be a topological group, (Y, d) a separable metric space, P a dense subset of X. Assume that X is locally compact (of the second category, respectively), $F : X \to P(Y)$ is $\mathcal{H}$-measurable ($\mathcal{B}$-measurable) satysfying one of the conditions

$$(i) \quad F \text{ is microperiodic multifunction,}$$

$$(ii) \quad P \text{ is countable and } F(px) \subset F(x) \text{ a.e. for every } p \in P.$$

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Theorem 5

Let $I \subset \mathbb{R}$ be a nontrivial interval, P a dense subset of $\mathbb{R}$. If $F : I \to P(\mathbb{R})$ is $\mathcal{L}$-measurable ($\mathcal{B}$-measurable, respectively) and there exists $E \in \mathcal{L}_0$ ($E \in \mathcal{B}_0$) such that

$$F(x+p) \subset F(x) \text{ for every } x \in I \setminus E, p \in P \text{ such that } x+p \in I,$$

then $\overline{F}$ is constant a.e..

Proposition 2 (Theorem 2 in [5])

Assume that X is a topological group, $P \subset X$ a dense set, (Y, d) is a metric space. Let $\varepsilon \geqslant 0$ and let $f : X \to Y$ satisfies

$$d(f(px), f(x)) \leqslant \varepsilon, \qquad x \in X, p \in P.$$

If f is continuous at some x_0, then

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Theorem 6

Assume that X is a topological group, $P \subset X$ a dense set, Y is a metric space. Let $\varepsilon \geqslant 0$ and let $F : X \to P(Y)$ satisfies

$$e(F(x), F(px)) \leqslant \varepsilon, \qquad x \in X, p \in P. \tag{1}$$

If F is h-continuous at some x_0, then

$$h(F(x), F(x_0)) \leqslant \varepsilon, \qquad x \in X.$$

Corollary 1

Assume that X is a topological group, $P \subset X$ a dense set, Y is a normed space. Let $C \subset Y$ and let $F : X \to P(Y)$ satisfies

$$F(px) \subset F(x) + C, \qquad x \in X, p \in P.$$

If F is h-continuous at some x_0 and there exists $y_0 \in Y$ and $\varepsilon > 0$ such that $C \subset y_0 + \varepsilon S$ (S is the closed unit ball in Y), then

$$h(F(x), F(x_0)) \leqslant \|y_0\| + \varepsilon, \qquad x \in X.$$

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Assume that X is a topological group, $P \subset X$ a dense set, Y is a normed space. Let $C \subset Y$ and let $f : X \to Y$ satisfies

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$$\|f(x) - f(x_0)\| \leqslant \|y_0\| + \varepsilon, \qquad x \in X.$$

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