Sets of ideal convergence of sequences of quasi-continuous functions

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Types of convergence of a sequence of reals

If $(y_n)_n$ is a sequence of reals then one of the following cases holds.

- $(y_n)_n$ is convergent (i.e., $\lim_n y_n \in \mathbb{R}$);
- $\lim_n y_n = -\infty$;
- $\lim_n y_n = \infty$;
- $-\infty < \underline{\lim}_n y_n < \overline{\lim}_n y_n < \infty$;
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Points of convergence of a sequence of functions

Let $\vec{f} = (f_n)_n \in \mathbb{R}^X$. Then X can be decomposed onto 7 sets:

$$
\begin{aligned}
E^1(\vec{f}) &= \{x\colon (f_n(x)) \text{ is convergent}\};\\
E^2(\vec{f}) &= \{x\colon \lim f_n(x) = -\infty\};\\
E^3(\vec{f}) &= \{x\colon \lim f_n(x) = +\infty\};\\
E^4(\vec{f}) &= \{x\colon -\infty < \underline{\lim}\, f_n(x) < \overline{\lim}\, f_n(x) < +\infty\};\\
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Sequences of continuous functions

Let X be a metric space.

Theorem (Hahn 1919, Sierpiński 1921)

$A = E^1(\vec{f})$ for some sequence $\vec{f}$ of continuous functions iff $A \in \mathcal{F}_{\sigma\delta}(X)$.

Kornfeld (1963) posed a problem of characterization of pairs $(E^2(\vec{f}), E^3(\vec{f}))$ for sequences $\vec{f}$ of continuous functions.

Theorem (Lipiński 1962)

For $(A, B) \subset X^2$ there exists a sequence $\vec{f}$ of continuous functions with $(A, B) = (E^2(\vec{f}), E^3(\vec{f}))$ iff

1. *$A, B \in F_{\sigma\delta}$;*
2. *there is $C \in G_\delta$ such that $A \subset C$ and $C \cap B = \emptyset$.*

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Sequences of continuous functions

Theorem (Lunina 1975)

For a sequence $(E^1, \dots E^7) \subset X^7$ there exists a sequence $\vec{f}$ of continuous functions with $(E^1, \dots, E^7) = (E^1(\vec{f}), \dots, E^7(\vec{f}))$ iff

1. $E^1, \dots, E^7$ *form a partition of* X;
2. E^1, E^2, E^3 *are* $F_{\sigma\delta}$ *in* X;
3. $E^2 \cup E^5 \cup E^7$ *and* $E^3 \cup E^6 \cup E^7$ *are* G_δ *in* X.

Let $\mathcal{F} \subset \mathbb{R}^X$ be a family of real-valued functions defined on a set X.

Definition

A sequence $(E^1, E^2, E^3) \subset X^3$ is called a Lipiński's triple for $\mathcal{F}$ if there is a sequence $\vec{f} = (f_n)_n \in \mathcal{F}$ such that $E^i = E^i(\vec{f})$ for $i = 1, 2, 3$.
The family of all Lipiński's triples for $\mathcal{F}$ is denoted by $\Lambda^3(\mathcal{F})$.

Definition

A sequence $(E^1, \ldots, E^7)$ is called a Lunina's 7-tuple for $\mathcal{F}$ if there is a sequence $\vec{f} = (f_n)_n \in \mathcal{F}$ such that $E^i = E^i(\vec{f})$ for $i = 1, \ldots, 7$.
The family of all Lunina's 7-tuples for $\mathcal{F}$ is denoted by $\Lambda^7(\mathcal{F})$.

Note that (E^1, E^2, E^3) is a Lipiński's triple for $\mathcal{F}$ iff it can be extended to some Lunina's 7-tuple for $\mathcal{F}$.

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Quasi-continuous functions

Definition

A function $f\colon X \to \mathbb{R}$ is quasi-continuous ($f \in \mathcal{QC}(X)$) if for each open set $V \subset \mathbb{R}$ the inverse-image of V is semi-open, i.e.

$$f^{-1}(V) \subset \mathrm{cl}(\mathrm{int} f^{-1}(V))$$

Note that $\mathcal{C}(X) \subset \mathcal{QC}(X)$ and (usually) $\mathcal{C}(X) \neq \mathcal{QC}(X)$.

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Lipiński's triples for $\mathcal{QC}(X)$

X – metric space.

Theorem (Wesołowska 2001)

$(E^1, E^2, E^3) \in \Lambda^3(\mathcal{QC}(X))$ *iff*

1. $E^i \cap E^j = \emptyset$ *for* $i \neq j$;
2. $E^i = (G_i \setminus P_i) \cup Q_i$, *where* G_i *are regular open,* P_i, Q_i *are meager in* X, $P_i \subset G_i$, $G_i \cap Q_i = \emptyset$; *and moreover*
3. $P_i \cap Q_j$ *are nowhere dense for all* $i, j \leq 3$, $i \neq j$.

Lunina's 7-tuples for $\mathcal{QC}(X)$

X – metric space.

Theorem

$(E^1, \ldots, E^7) \in \Lambda^7(\mathcal{QC}(X))$ iff

1. *$E^1, \ldots E^7$ are a partition of X;*
2. *$E^i = (G_i \setminus P_i) \cup Q_i$, where G_i is regular open, P_i, Q_i are meager in X, $P_i \subset G_i$, $G_i \cap Q_i = \emptyset$ for all $i = 1, \ldots, 7$; and moreover*
3. *$P_i \cap Q_j$ are nowhere dense for $(i, j) \in (\{1, 4\} \times \{2, 3, 5, 6, 7\}) \cup (\{2, 5\} \times \{3, 6, 7\}) \cup (\{3, 6\} \times \{2, 5, 7\})$.*

Sets of ideal convergence

Let $\mathcal{I}$ be an ideal on ω. Let $\vec{f} = (f_n)_n \in \mathbb{R}^X$. We define 7 types of sets of $\mathcal{I}$-convergence and divergence of the sequence $\vec{f}$.

$$
\begin{aligned}
E^1_{\mathcal{I}}(\vec{f}) &= \{x\colon (f_n(x)) \text{ is } \mathcal{I}\text{-convergent}\};\\
E^2_{\mathcal{I}}(\vec{f}) &= \{x\colon \mathcal{I}-\lim f_n(x) = -\infty\};\\
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Let $\mathcal{I}$ be an ideal on ω and $\mathcal{F} \subset \mathbb{R}^X$.

Definition

A sequence $(E^1, \ldots, E^7)$ is called an $\mathcal{I}$-Lunina's 7-tuple for $\mathcal{F}$ if there is a sequence $\vec{f} = (f_n)_n \in \mathcal{F}$ such that $E^i = E^i_{\mathcal{I}}(\vec{f})$ for $i = 1, \ldots, 7$.
The family of all $\mathcal{I}$-Lunina's 7-tuples for $\mathcal{F}$ is denoted by $\Lambda^7_{\mathcal{I}}(\mathcal{F})$.

In a similar way we define the family $\Lambda^3_{\mathcal{I}}(\mathcal{F})$ of $\mathcal{I}$-Lipiński's triples for $\mathcal{F}$.

Ideal Lipiński's triples and Lunina's 7-tuples

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Ideal Lunina's 7-tuples for $C(X)$

Theorem (Borzystowski and Recław (2009))

Let X be a metric space and $\mathcal{I}$ be an F_σ ideal. Then $\Lambda^7_{\mathcal{I}}(\mathcal{C}(X)) = \Lambda^7(\mathcal{C}(X))$.

An analogous result does not hold for quasi-continuous functions!

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Let X be a metric space and $\mathcal{I}$ be an F_σ ideal. Then $\Lambda^7_{\mathcal{I}}(\mathcal{C}(X)) = \Lambda^7(\mathcal{C}(X))$.

An analogous result does not hold for quasi-continuous functions!

The game $G(\mathcal{I})$ is defined as follows:

- In the n'th move Player I plays a set $A_n \in \mathcal{I}$;
- Then Player II plays an $a_n \in \omega \setminus A_n$.

Player I wins if $\{a_n : n \in \omega\} \in \mathcal{I}$.

Theorem (TN, Szuca)

If $\mathcal{I}$ is Borel ideal then the game $G(\mathcal{I})$ is determined.

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Let X be a metric space and $\mathcal{I}$ be an F_σ ideal.

Theorem

If Player II has a winning strategy in $G(\mathcal{I})$ then $\Lambda^3_{\mathcal{I}}(\mathcal{QC}) = \Lambda^3(\mathcal{QC})$.

Theorem

If Payer I has a winning strategy in $G(\mathcal{I})$ then $(E^1, E^2, E^3) \in \Lambda^3_{\mathcal{I}}(\mathcal{QC})$ iff E^1, E^2 E^3 are pairwise disjoint and have the Baire property.

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Ideal Lunina's 7-tuples for $\mathcal{QC}(X)$

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If Payer I has a winning strategy in $G(\mathcal{I})$ then $(E^1, \ldots, E^7) \in \Lambda^7_{\mathcal{I}}(\mathcal{QC})$ if $E^1, \ldots, E^7$ forms a partition of X onto sets with the Baire property.

Theorem

If $\mathcal{I}$ is c.g. ideal then $\Lambda^7_{\mathcal{I}}(\mathcal{QC}) = \Lambda^7(\mathcal{QC})$.

Remark

There exists an F_σ ideal $\mathcal{I}$ for which Player II has a winning strategy in $G(\mathcal{I})$ and $\Lambda^7_{\mathcal{I}}(\mathcal{QC}) \neq \Lambda^7(\mathcal{QC})$.

Problem

Characterize Borel ideals (F_σ ideals) $\mathcal{I}$ for which the condition $\Lambda^7_{\mathcal{I}}(\mathcal{QC}(X)) = \Lambda^7(\mathcal{QC}(X))$ holds for any metric space X.

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Ideal Lunina's 7-tuples for $\mathcal{QC}(X)$

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Appendix: Ideal convergence

Let $\mathcal{I}$ be an ideal of subsets of ω.

Definition

A sequence $(x_n)_{n\in\omega}$ of reals is

- $\mathcal{I}$-convergent to $x \in \mathbb{R}$ ($\mathcal{I} - \lim x_n = x$) iff

$$\forall_{\varepsilon>0} \{n \in \omega : |x_n - x| \geq \varepsilon\} \in \mathcal{I}.$$

- $\mathcal{I} - \lim x_n = +\infty$ iff $\{n \in \omega : \ x_n < M\} \in \mathcal{I}$ for any $M \in \mathbb{R}$;
- $\mathcal{I} - \lim x_n = -\infty$ iff $\{n \in \omega : \ x_n > M\} \in \mathcal{I}$ for any $M \in \mathbb{R}$;
- $\mathcal{I} - \overline{\lim}\, x_n = \inf \{\alpha \colon \{n \colon x_n > \alpha\} \in \mathcal{I}\}$;
- $\mathcal{I} - \underline{\lim}\, x_n = \sup \{\alpha \colon \{n \colon x_n < \alpha\} \in \mathcal{I}\}$.

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- $\mathcal{I} - \lim x_n = +\infty$ iff $\{n \in \omega : \ x_n < M\} \in \mathcal{I}$ for any $M \in \mathbb{R}$;
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Let $\mathcal{I}$ be an ideal of subsets of ω.

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Appendix: Ideal convergence

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Appendix: Laflamme's Theorem

Theorem (Laflamme, 1996)

1. *Player I has a winning strategy if and only if $\mathcal{I}^\star$ is not weakly Ramsey;*
2. *Player II has a winning strategy if and only if $\mathcal{I}^\star$ is ω-$+$-diagonalizable.*

Definition

$\mathcal{I}^\star$ is ω-$+$-diagonalizable if there are sets $\{X_n \in \mathcal{I}^+ : n \in \omega\}$ such that for each $F \in \mathcal{I}^\star$ there is $n \in \omega$ with $X_n \subset F$.

Definition

$\mathcal{I}^\star$ is weakly Ramsey if any $\mathcal{I}^\star$-tree has a branch in $\mathcal{I}^+$.

Definition

A tree $\mathcal{T} \subset \omega^{<\omega}$ is an $\mathcal{I}^\star$-tree if for each $s \in \mathcal{T}$ there is an $\mathcal{X}_s \in \mathcal{I}^\star$ such that $s \frown n \in \mathcal{T}$ for all $n \in \mathcal{X}_s$.

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Appendix: Countably generated ideals

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An ideal $\mathcal{I}$ is countably generated if there is a a family $\{I_n \colon n < \omega\}$ of sets from $\mathcal{I}$ such that for each $I \in \mathcal{I}$ there is $n < \omega$ with $I \subset I_n$.

Theorem (Farah (2000))

Every c.g. ideal is isomorphic either to FIN or to the ideal $FIN \times \emptyset = \{A \subset \mathbb{N} \times \mathbb{N} \colon \{n \colon A \cap (\{n\} \times \mathbb{N}) \neq \emptyset\} \in FIN\}$.

Corollary

Let $\mathcal{I}$ be a c.g. ideal. Then

- *$\mathcal{I}$ is of the type F_σ.*
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