Beyond the sets of subsums

Franciszek Prus-Wiśniowski

Institute of Mathematics
Szczecin University
Poland

XXVIII International Summer Conference on Real Functions Theory, Stara Lesna 2014

H. Steinhaus, *Nowa własność monogości Cantora*, Wektor (1917) 1-3 [English translation: H. Steinhaus, *Selected papers*, PWN, Warszawa 1985 (pages 205-207)]

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$$C + C = [0, 2]$$

Introduction

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W.R. Utz, *The distance set for the Cantor discontinuum*, Amer. Math. Monthly 58(1951) 407-408

Theorem

$C + \lambda C$ *is an interval iff* $\frac{1}{3} \leq |\lambda| \leq 3.$

M. Pawłowicz, *Linear combinations of the classic Cantor set*, Tatra Mountain Math. Publ. 56(2013) 47-60

Theorem

Given a $\lambda \in (0, 1)$, *let* $n := \max\{ k \in \mathbb{N}_0 : \ \lambda < \frac{1}{3^k} \}$ *and let*

$$I_n \ = \ \bigsqcup_{i=1}^{2^n} [a_i, \ b_i]$$

be the n-th iteration of the classic construction of the Cantor set. Then

$$C \ + \ \lambda \, C \ = \ \bigsqcup_{i=1}^{2^n} [a_i, \ b_i + \lambda].$$

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a Cantor set $=$ a bounded, perfect and nowhere dense subset of $\mathbb{R}$

Regular Cantor sets (or dynamically defined Cantor sets):

$$\lambda \; = \; (\lambda_0, \, \lambda_1', \, \lambda_1, \, ..., \, \lambda_k', \, \lambda_k) \; \in \; \mathbb{R}^{2k+1} \qquad \text{a probability vector}$$

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The family $\{P_i : \; i = 0, 1, \ldots, k\}$ is called **the Markov partition** for C.

$$\phi : \bigcup_{i=0}^{k} P_i \;\to\; [0, 1] \qquad \text{a } C^r\text{-function}$$

Each $\phi_i \;:=\; \phi\big|_{P_i}$ is a surjective expanding map, that is,
$|\phi_i(x) - \phi_i(y)| \;>\; \alpha|x - y|$ for some $\alpha > 1$. Then

$$C \;=\; C(\lambda, \phi) \;:=\; \bigcap_{n\in\mathbb{N}} \phi^{-n}(P_0 \cup \ldots \cup P_k)$$

is a C^r**-regular Cantor set**.

If each ϕ_i is an affine map, then we say that $C = C(\lambda, \phi)$ is an **affine Cantor set**.

B. Honary, C.G. Moreira, M. Pourbarat, *Stable intersections of affine Cantor sets*, Bull. Braz. Math. Soc. New Series 36(3)(2005) 363-378

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J. Palis, *Homoclinic orbits, hyperbolic dynamics and dimension of Cantor sets*, Contemp. Math. 58(1987) 203-216

Palis' Question(s)

Is it true that for regular (affine) Cantor sets C_1, C_2 *we have either* $\mu(C_1 + C_2) = 0$ *or else* $C_1 + C_2$ *contains an interval ?*

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Actually, what is of interest to the dynamical systems people is when a Cantor set C_1 intersects a translate of another Cantor set C_2.

$$\{x : C_1 \cap (x + C_2) \neq \emptyset\} = C_1 - C_2 = C_1 + (-C_2)$$

The **central Cantor sets**:

$$C \;=\; \bigcap_{b\in\mathbb{N}} I_n$$

such that there exists a sequence $(q_n) \in (0, \frac{1}{2})$ where $I_0 = [0, 1]$ and

$$I_n \;=\; \bigsqcup_{i=1}^{2^n} P_i^{(n)} \qquad\qquad (\, P_i^{(n)} \ - \text{closed intervals})$$

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The open intervals removed from $P_i^{(n)}$ in each step of the construction are central with respect to $P_i^{(n)}$, that is, they share the midpoint with $P_i^{(n)}$.

$$\frac{|P_{2i}^{(n+1)}|}{|P_i^{(n)}|} \;=\; \frac{|P_{2i-1}^{(n+1)}|}{|P_i^{(n)}|} \;=\; q_{n+1} \qquad \text{for } \; n \in \mathbb{N}_0,\;\; i = 1,\, \ldots,\, 2^n$$

q_n = the nth **ratio of retention (ratio of dissection)**. The corresponding central Cantor set is then denoted by $C \;=\; C_{[(q_n)]}$.

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$$\frac{|P_{2i}^{(n+1)}|}{|P_i^{(n)}|} = \frac{|P_{2i-1}^{(n+1)}|}{|P_i^{(n)}|} = q_{n+1} \qquad \text{for } n \in \mathbb{N}_0,\ i = 1, \ldots, 2^n$$

q_n = the nth **ratio of retention (ratio of dissection)**. The corresponding central Cantor set is then denoted by $C = C_{[(q_n)]}$. If $q_n \equiv \alpha$, we will write $C = C_{[\alpha]}$. The classic Cantor set is $C_{classic} = C_{[\frac{1}{3}]}$.

R. Bamón, S. Plaza, J. Vera, *On central Cantor sets with self-arithmetic difference of positive Lebesgue measure* J. London Math. Soc. 52 (1995) 137-146

Theorem

Every central Cantor set is regular of class C^0.

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J. Palis knew the following facts

$$\begin{aligned} C_{[\alpha]} + C_{[\alpha]} &= [0, 2] \qquad \text{for } \alpha \in [\tfrac{1}{3}, \tfrac{1}{2}) \\ \mu\left(C_{[\alpha]} + C_{[\alpha]}\right) &= 0 \qquad \text{for } \alpha \in (0, \tfrac{1}{3}) \end{aligned}$$

He wrote: *we do not know the answer in general for* $C_{[\alpha]} + C_{[\beta]}$.

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J. Palis, F. Takens, *Cycles and measure of bifurcation sets for two dimensional diffeomorphisms*, Invent. Math. 82 (1985) 397-422

Theorem

Let C_1, C_2 *be* C^r*-regular Cantor sets with* $r \geq 1$. *If* $dim_H C_1 + dim_H C_2 < 1$, *then* $\mu(C_1 + C_2) = 0$.

J. Marstand, *Some fundamental geometric properties of plane sets of fractional dimensions*, Proc. London Math. Soc. 4 (1952) 257-302

Theorem

Let C_1, C_2 *be* C^r*-regular Cantor sets with* $r \geq 1$*. If* $dim_H C_1 + dim_H C_2 > 1$*, then* $\mu(C_1 + \lambda C_2) > 0$ *for almost all* $\lambda \in \mathbb{R}$*.*

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Theorem

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The last two theorems remain true even if we drop any regularity conditions on the Cantor sets C_1, C_2, but then we have to replace the Hausdorff dimension $\dim_H$ by the limit capacity

$$d(A) := \limsup_{\epsilon \to 0+} \frac{\ln N(\epsilon)}{-\ln \epsilon}$$

where $N(\epsilon)$ is the minimal number of ϵ-balls needed to cover the set A. In general, $d(A) \geq \dim_H A$, but the equality holds for C^1-regular Cantor sets. Recall that for $\lambda = 1$ we get $\mu(C_{[\alpha]} + \lambda C_{[\alpha]}) = 0$ for $\frac{1}{4} < \alpha < \frac{1}{3}$.

Since $\dim_H C_{[\alpha]} = \dfrac{\ln 2}{-\ln \alpha}$, we have $\dim_H C_{[\alpha]} + \dim_H C_{[\alpha]} > 1$ for those α's

and therefore $\lambda = 1$ is an exceptional value for the Marstand theorem (for the pair $C_{[\alpha]}$, $C_{[\alpha]}$).

A. Sannami, *An example of a regular Cantor set whose difference set is a Cantor set with positive Lebesgue measure*, Hokkaido Math. J. 21 (1992) 7-24

Theorem

If $q_i < \frac{1}{3}$ *for all* i, *then* $\mu C_{[(q_i)]} = 0$ *and* $C_{[(q_i)]} + C_{[(q_i)]}$ *is a Cantor set. However, if* $\prod_{i=1}^{\infty}(3q_i) > 0$, *then* $\mu\left(C_{[(q_i)]} + C_{[(q_i)]}\right) > 0$.

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Bamón, Plaza and Vera found a very handy analytic characterization of a large subclass of C^r-regular central Cantor sets.

Theorem

Let $r \geq 1$ and let (q_i) be a sequence of numbers from $(0, \frac{1}{2})$ satisfying the following conditions:

(i) *there exists a limit $q := \lim_{i\to\infty} q_i \in (0, \frac{1}{2})$;*

(ii) *eiter $q_i < q$ for all i or $q_i > q$ for all i.*

Then $C_{[(q_i)]}$ is regular of class C^r if and only if

$$\lim_{n\to\infty} \frac{1 - \frac{q_n}{q}}{\prod_{i=1}^{n-1} q_i^{r-1}} = 0.$$

Using the characterization, they found infinitely many counterexamples to the Palis conjecture.

Theorem

For every $r \in \mathbb{N}$ the class of central Cantor sets that are C^r but not C^{r+1}, that have zero Lebesgue measure and whose self-arithmetic sums are Cantor sets with postive Lebesgue measure, is non-empty.

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F. Prus-Wiśniowski, unpublished

Theorem

Every central Cantor set is the arithmetic sum of two central Cantor sets of Lebesgue measure zero.

The theorem is elegant, but it does not contradict the Palis conjecture, because it says nothing about regularity of the sets.

F. Prus-Wiśniowski, unpublished

Theorem

Let $r \in \mathbb{N}$ *and let* $C = C_{[(q_i)]}$ *be a* C^r*-regular central Cantor set satisfying the conditions:*

(i) *there exists a limit* $q := \lim_{i\to\infty} q_i \in (0, \frac{1}{2})$ *;*

(ii) *the sequence* (q_i) *is monotonic, but not eventually constant.*

Then C *is the arithmetic sum of two* C^r*-regular central Cantor sets of Lebesque measure zero.*

P. Larsson, *L'ensemble différence de deux ensembles de Cantor aléatoires*, C.R. Acad. Sci. Paris 310 (1990) 735-738

P. Larsson defined a special family of random Cantor sets and, using methods from the theory of branching processes, proved that for almost all pairs C_1, C_2 of such sets if $\dim_H C_1 = \dim_H C_2 > \frac{1}{2}$ (which implies $\dim_H C_1 + \dim_H C_2 > 1$), then $C_1 + C_2$ contains an interval.
Although the main idea of Larsson's argument is brilliant, unfortunately, the proof contains significant gaps and even incorrect reasoning. The faults of Larsson's work have been pinpointed recently and a correct proof was supplied in
M. Dekking, K. Simon, B. Székely, *The algebraic difference of two random Cantor sets: the Larsson family*, Ann. Prob. 39 (2) (2011) 549-586

C.G. Moreira, J.- C. Yoccoz, *Stable intersections of regular Cantor sets with large Hausdorff dimensions*, Ann. Math. 154 (2001) 45-96

$$\Omega^\infty = \{\, (C_1,\, C_2) : \ C_1,\, C_2 \text{ are } C^r\text{-regular Cantor sets and } \dim_H C_1 + \dim_H C_2 > 1 \,\}$$

Theorem

For $r > 1$ *there is an open and dense set* $U \subset \Omega^\infty$ *such that if* $(C_1,\, C_2) \in U$*, then the interior of* $C_1 + C_2$ *is dense in* $C_1 + C_2$*.*

Let $\sum a_n$ be an absolutely convergent series of real numbers with at least one non-zero term. The **set of subsums** is then

$$E(a_n) = E(\sum a_n) := \left\{ x \in \mathbb{R} : \exists\, B \subset \mathbb{N} \sum_{n \in B} a_n = x \right\}.$$

S. Kakeya, *On the partial sums of an infinite series*, Tohoku Sci. Rep. 3 (1914) 159-164

Theorem

The set of subsums of an absolutely convergent series is bounded and perfect.

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S. Kakeya, *On the partial sums of an infinite series*, Tohoku Sci. Rep. 3 (1914) 159-164

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The set of subsums of an absolutely convergent series is bounded and perfect.

From now on, we will usually assume that $\sum a_n \;=\; 1$ and $a_n \geq a_{n=1} > 0$ for all n. In that case $E(a_n) \subset [0, 1]$.

F. Prus-Wiśniowski, *Beyond the sets of subsums*, preprints of the Faculty of Mathematics and Informatics, Łódź University 2013 (`www.mat.uni.lodz.pl/preprints,all.html`)

the n-th remainder $\qquad r_n \;=\; \sum_{i>n} a_i, \;\; n \in \mathbb{N}_0$

Theorem

$$E(a_n) \;=\; [0,1] \;\; \textit{iff} \;\; r_n \geq a_n \;\; \textit{for all } n.$$

Theorem

If $a_n < r_n$ *for all sufficiently large* n*, then* $E(a_n)$ *is a Cantor set. In that case* $\mu E(a_n) = \lim_{n\to\infty} 2^n r_n$.

Theorem

$E(a_n)$ *is a union of finitely many closed and bounded intervals if and only if* $r_n \geq a_n$ *for all sufficiently large* n.

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A.D. Weinstein, B.E. Shapiro, *On the structure of the set of $\overline{\alpha}$-representable numbers*, Izv. Vyssh. Uchebn. Zaved. Mat. 24(1980) 8-11

Cz. Ferens, *On the range of purely atomic measures*, Studia Math. 77(1984) 261-263

A bounded set $A \subset \mathbb{R}$ is said to be an **M-Cantorval** if it is perfect and all gaps of A (including the external ones) are on each (finite) side accumulated by infinitely many intervals and by infinitely many gaps.

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Theorem

Any two M-Cantorvals are homeomorphic.

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Theorem

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The Gutrie-Nymann-Sáenz Classification Theorem

Let $\sum a_n$ be an absolutely convergent series of real terms. Then exaxtly one of the following cases holds:

(i) $E(a_n)$ *is a finite set;*

(ii) $E(a_n)$ *is the union of a finite family of bounded and closed intervals;*

(iii) $E(a_n)$ *is a Cantor set;*

(iv) $E(a_n)$ *is an M-Cantorval.*

Z. Nitecki, *Subsum sets: intervals, Cantor sets, and Cantorvals*, arXiv: 1106.3779v1[math.HO]

Theorem

Some Kenyon sets are M-Cantorvals.

Z. Nitecki, *Subsum sets: intervals, Cantor sets, and Cantorvals*, arXiv: 1106.3779v1[math.HO]

Theorem

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Let $\{k_1, \ldots, k_m\}$ be a finite set of reals and let $q \in (0, 1)$. The series

$$\sum(k_1, \ldots, k_m; q) := \sum_{\substack{i \in \mathbb{N} \\ j \in \{j, \ldots, m\}}} k_j q^i$$

is called a **multigeometric series**.

A. Bartoszewicz, M. Filipczak, E. Szymonik, *Multigeometric sequences and Canotrvals*, Central European J. Math. 12(7)(2014) 1000-1007

Theorem

Let $k_1 \geq k_2 \geq \ldots \geq k_m$ be positive integers and $K := \sum_{i=1}^m k_i$. Assume that there exists positive integers n_0 and n such that each of the numbers $n_0,\ n_0+1,\ \ldots,\ n_0+n$ can be obtained by summing up elements of a subset of $\{k_1, \ldots, k_m\}$. If

$$\frac{1}{n+1} < q \leq \frac{k_m}{K+k_m},$$

then $E\big(\sum(k_1, \ldots, k_m;\ q)\big)$ is an M-Cantorval.

Theorem

For any $k \in \mathbb{N}$, the set of subsums

$$E\Big(\sum(3, \underbrace{2, \ldots, 2}_{k\ \ times};\ \tfrac{1}{2k+2})\Big)$$

is an M-Cantorval.

M. Banakiewicz, F. Prus-Wiśniowski, *M-Cantorvals of Ferens type*, under review in the Bull. London Math. Soc. [for 6 months now and going...]

A multigeometric series of the form $\sum(m+k-1,\, m+k-2,\, \dots,\, m;\, q)$ where k, m are positive integers such that $k \geq m+1$, is said to be **of Ferens type**. Given such a series, we set $s := m + (m+1) + \cdots + (m+k-1)$.

Theorem

Let $\Sigma = \sum(m+k-1,\, m+k-2,\, \dots,\, m;\, q)$ *be a Ferens type series.*

(i) If $0 < q < \frac{1}{s+1}$, *then* $E(\Sigma)$ *is a Cantor set.*

(ii) If $\frac{1}{s-2m+1} \leq q < \frac{m}{s+m}$, *then* $E(\Sigma)$ *is an M-Cantorval. In this case*

$$\mu E \quad = \quad (s-2m)\,\frac{q}{1-3q} \quad = \quad \mu(int\, E).$$

(iii) If $\frac{m}{s+m} \leq q < 1$, *then* $E(\Sigma) \;=\; [0,\, \frac{sq}{1-q}]$.

In the past year, Bartoszewicz, Filipczak and Szymonik together with Taras Banakh strenghtened the results on sets of subsums of multigeometric series significantly.

T. Banakh, A. Bartoszewicz, M. Filipczak, E. Szymonik, *Topological and measure properties of some self-similar sets*, arXiv: 1403.0098v1[math.GN]

Using deep results of Boris Solomyak on the distribution of random multigeometric series, they managed to show that for almost all q in the mysterious interval, the set of subsums $E\left(\sum(k_1, \ldots, k_m; q)\right)$ has positive Lebesgue measure. On the other hand, under some mild conditions, they showed that there exists a sequence of ratios $(q_i)_{i\in\mathbb{N}}$ decreasing to the left end of the mysterious interval such that $\mu\left(E(\sum(k_1, \ldots, k_m; q)\right) = 0$ for all i.

Now, let $C_{[(q_i)]}$ be a central Cantor set with $[0, 1]$ as the fudamental interval. In particular, all $q_i \in (0, \frac{1}{2})$. Setting

$$a_n := (1 - q_n) \prod_{i=1}^{n-1} q_i \qquad \text{for } n \in \mathbb{N} \qquad \left(\prod_{i=1}^{0} q_i := 1 \right),$$

we obtain a series $\sum a_n = 1$ of positive terms such that

$$C_{[(q_i)]} = E(\sum a_n).$$

Moreover, $q_i = r_i / r_{i-1}$ for all $i \in \mathbb{N}$. In particular,

$$C_{classic} = C_{[\frac{1}{3}]} = E(\sum \frac{2}{3^n}).$$

Now, let $C_{[(q_i)]}$ be a central Cantor set with $[0, 1]$ as the fudamental interval. In particular, all $q_i \in (0, \frac{1}{2})$. Setting

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J. Nymann, *Linear combination of Cantor sets*, Colloq. Math. 68(1995) 259-284

He investigated the topological type of a particular linear combination

$$E_q + \lambda E_q \qquad \text{for } \lambda \in (0, 1] \text{ and } q \in (0, \frac{1}{2})$$

where

$$E_q := E(\sum_{n=1}^{\infty} q^n).$$

In particular, $C_{classic} = 2E_{\frac{1}{3}}$ and, in general, $C_{[q]} = (\frac{1}{q} - 1)E_q$. On the other hand,

$$E_q + \lambda E_q = E\big(\Sigma(1, \lambda; q)\big).$$

In particular, $C_{classic} = 2E_{\frac{1}{3}}$ and, in general, $C_{[q]} = (\frac{1}{q} - 1)E_q$. On the other hand,

$$E_q + \lambda E_q = E\big(\Sigma(1, \lambda; q)\big).$$

Theorem

(i) If $q \in [\frac{1}{3}, \frac{1}{2})$, *then* $E_q + \lambda E_q$ *is the union of a finite family of closed bounded intervals for every* $\lambda \in (0, 1]$.

(ii) If $q \in [\frac{1}{4}, \frac{1}{3})$, *then* $E_q + \lambda E_q$ *is the union of a finite family of closed bounded intervals iff*

$$\lambda \in \left[q^k(1-2q), \frac{q^{k+1}}{1-2q}\right] \qquad \text{for some } k \in \mathbb{N}_0.$$

(iii) $E_{\frac{1}{4}} + \frac{2}{3 \cdot 4^n} E_{\frac{1}{4}}$ *is an M-Cantorval for every* $n \in \mathbb{N}$.

(iv) If $q \in (0, \frac{1}{4})$, *then for all* $\lambda \in (0, 1]$ *the set* $E_q + \lambda E_q$ *is either a Cantor set or an M-Cantorval.*

A Cantor set is said to **homogeneous** if it is affine and all intervals of its Markov partition have the same length.

A perfect subset of $\mathbb{R}$ such that any gap of it has an interval adjacent to its right (left) endpoint and is accumulated on the left (right) endpoint by infinitely many intervals and gaps, is called an **L-Cantorval** (**R-Cantorval**).

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P. Mendes, F. Oliveira, *On the topological structure of the arithmetic sum of two Cantor sets*, Nonlinearity 7(1994) 329-343

Theorem

If C_1 and C_2 are homogeneous Cantor set with fundamental intervals of equal length (assume $[0, 1]$), then exactly one of the following cases holds:

(i) $C_1 + C_2 = [0, 2]$;

(ii) $C_1 + C_2$ *is a Cantor set;*

(iii) $C_1 + C_2$ *is an R-Cantorval such that* $[0, 1]$ *is contained in one of the intervals of* $C_1 + C_2$*;*

(iv) $C_1 + C_2$ *is an L-Cantorval such that* $[1, 2]$ *is contained in one of the intervals of* $C_1 + C_2$*;*

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If both C_1 and C_2 are symmetric, then only the possibilities (i), (ii) and (v) can occur.

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The assumption of equal length is indispensable:
R. Anisca, Ch. Chlebovec, *On the structure of arithmetic sums of Cantor sets with constant ratios of dissection*, Nonlinearity 22(2009) 2127-2140

Thickness of Cantor sets

C – a Cantor set; G – an internal gap of C;

A closed and bounded interval B is called a **span of a gap** G if it is the bounded interval obtained from $\mathbb{R}$ by removing the gap G and any other gap (possibly an external one).
The **bridge** B of the gap G is then the longest among all spans of G containing only gaps shorter than G. A bridge might not be unique, but its length is uniquely determined. We refer to (B, G) as a bridge/gap pair. The thickness of the Cantor set C is then defined as

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J. Palis, F. Takens, *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations*, Cambridge Stud. Adv. Math. 35, Cambridge Univ. Press, 1995

R.L. Kraft, *Intersections of thick Cantor sets*, Memoirs Amer. Math. Soc. 97(1992)

S.E. Newhouse, *Lectures on dynamical systems*, Dynamical Systems, CIME Lectures, Birkh auser Verlag 1980 (pp. 1-114)

The Newhouse Gap Lemma

Let C_1 and C_2 be Cantor sets with fundamental intervals I_1, I_2 such that

(i) *I_1 is longer than each internal gap of C_2;*

(ii) *I_2 is longer than each internal gap of C_1.*

If $\tau C_1 \cdot \tau C_2 \geq 1$, then $C_1 + C_2 = I_1 + I_2$.

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$$\tau C_{[\alpha]} = \frac{\alpha}{1-2\alpha} \qquad \text{for } \alpha \in (0, \tfrac{1}{2}).$$

The inequality $\tau C_{[\alpha]} \cdot \tau C_{[\beta]} \geq 1$ is equivalent to

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Refined thickness methods were studied and applied in: S. Astels, *Cantor sets and numbers with restricted partial quotients*, Trans. Amer. Math. Soc. 352(2000) 133-170

Among others he generalized the Newhouse Gap Lemma.

J. Palis: *We do not know in general what is* $C_{[\alpha]} + C_{[\beta]}$.

The problem is utterly difficult and only some more specialized questions have been answered so far. Mendes and Oliveira in their 1994 paper examined the issue of when $C_{[\alpha]} + C_{[\beta]} = [0, 2]$.

We may assume that $0 < \beta \leq \alpha < \frac{1}{2}$ by symmetry. The triangle of vertices $(0, 0)$, $(\frac{1}{2}, 0)$ and $(\frac{1}{2},, \frac{1}{2})$ is foliated by the curves $\beta = \alpha^t$, $t \geq 1$.

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For each $t \geq 1$, there is a unique $\alpha_2(t) \in (0, \frac{1}{2})$ such that $(\alpha_2(t), \alpha_2(t)^t)$ belongs to the curve $\frac{\alpha}{1-\alpha} + \frac{\beta}{1-\beta} = 1$.

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This equation means that the product of the thickness of $C_{[\alpha]}$ and $C_{[\beta]}$ is equal to one, so that for all pairs (α, β) above the hyperbola the sum $C_{[\alpha]} + C_{[\beta]}$ is the interval $[0, 2]$ by the Newhouse Gap Lemma.

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Further, for each $t \geq 1$, there is a unique $\alpha_1(t) \in (0, \frac{1}{2})$ such that $(\alpha_1(t), \alpha_1(t)^t)$ belongs to the line $\beta = 1 - 2\alpha$. Moreover, $\alpha_1(t) < \alpha_2(t)$ for all $t > 1$ and $\alpha_1(1) = \alpha_2(1) = \frac{1}{3}$.

P. Mendes, F. Oliveira, *On the topological structure of the arithmetic sum of two Cantor sets*, Nonlinearity 7(1994) 329-343

Theorem

For each $t \geq 1$ *there is a unique* $\alpha(t) \in [\alpha_1(t), \alpha_2(t)]$ *such that*

$$C_{[\alpha]} + C_{[\beta]} = [0, 2] \qquad \textit{if and only if} \qquad \alpha \geq \alpha(t).$$

Moreover,

(i) *if t is irrational, then* $\alpha(t) = \alpha_2(t)$*;*

(ii) *if* $t = n$ *or* $t = \frac{n+1}{n}$*, then* $\alpha(t) = \alpha_1(t)$.

C.A. Cabrelli, K.E. Hare, U.Molters, *Sums of Cantor sets yielding an interval*, J. Australian Math. Soc. 73(2002) 405-418

Theorem

Let $0 < \beta < \alpha < \frac{1}{2}$. *Then the following statements are equivalent:*

(i) $C_{[\alpha]} + C_{[\beta]} = [0, 2]$;

(ii) *For every* $m, n \in \mathbb{N}$ *at least one of the following inequalities holds:*

(a) $\beta^{n-1}(1-2\beta) \le \alpha^m$;
(b) $\alpha^{m-1}(1-2\alpha) \le \beta^n$.

(iii) *There are no positive integers* $n, , m$ *such that*

$$\ln(1-2\alpha) < (n+1)\ln\beta - (m-1)\ln\alpha < \ln\frac{1}{1-2\beta}.$$

$C_{[\alpha]} + C_{[\beta]}$

Corollary

If $\frac{\ln \beta}{\ln \alpha}$ *is irrational, then*

$$C_{[\alpha]} + C_{[\beta]} = [0, 2] \qquad \textit{if and only if} \qquad \tau C_{[\alpha]} \cdot \tau C_{[\beta]} \geq 1.$$

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Let $\beta = 1 - 2\alpha$. *Then*

$$C_{[\alpha]} + C_{[\beta]} = [0, 2] \qquad \textit{if and only if} \qquad \frac{\ln\beta}{\ln\alpha} \in \mathbb{N} \cup \left(1 + \frac{1}{\mathbb{N}}\right).$$

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Here is one of their hypotheses: for every rational value of $\frac{\ln\beta}{\ln\alpha}$ there is a choice of α such that $\tau C_{[\alpha]} \cdot \tau C_{[\beta]} < 1$ and $C_{[\alpha]} + C_{[\beta]} = [0, 2]$.

They proved it for $\frac{\ln\beta}{\ln\alpha} = 1 + \frac{p}{q}$, $p, q \in \mathbb{N}$ and relatively prime, $p \leq 8$, and also they proved it for all $\frac{\ln\beta}{\ln\alpha} \in \mathbb{N} + \frac{1}{2}$.

B. Solomyak, *On the measure of arithmetic sums of Cantor sets*, Indag. Mat. 8(1)(1997) 133-141

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$$\dim_H\left(C_{[\alpha]} + C_{[\beta]}\right) \leq \dim_H C_{[\alpha]} + \dim_H C_{[\beta]} = \frac{\ln 2}{-\ln \alpha} + \frac{\ln 2}{\ln \beta} < 1$$

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For any $\alpha \in (0, \frac{1}{2})$ there is a set $W_\alpha \subset (0, \frac{1}{2})$ of zero Lebesgue measure such that

$$dim_H C_{[\alpha]} + dim_H C_{[\beta]} > 1 \qquad \textit{implies} \qquad \mu\big(C_{[\alpha]} + C_{[\beta]}\big) > 0$$

for all $\beta \in (0, \frac{1}{2}) \setminus W_\alpha$.

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Y. Peres, P. Shmerkin, *Resonanse between Cantor sets*, Ergodic Theory Dyn, Syst. 29(1)(2009) 201-221

Theorem

If $\frac{\ln\beta}{\ln\alpha}$ *is irrational, then*

$$dim_H\big(C_{[\alpha]} + C_{[\beta]}\big) = \min\big\{1, dim_H C_{[\alpha]} + dim_H C_{[\beta]}\big\}.$$

Hey, you in the last row,

wake up!

Thank you for attention