Topological and measure properties of some self-similar sets.

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Definition

Set $E(x)$

Suppose that $x = (x(0), x(1), x(2), \dots)$ and $x \in l_1 \setminus c_{00}$ then

$$E(x) = \left\{ \sum_{n=0}^{\infty} \varepsilon_n x(n) : \varepsilon_n \in \{0,1\} \right\}$$

is the set of all subsums of the series $\sum_{n=0}^{\infty} x(n)$, called the achievement set of x.

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The following properties of sets $E(x)$ were described in 1914 by S. Kakeya:

I. $E(x)$ is a compact perfect set.

II. If $|x(n)| > \sum_{i>n} |x(i)|$ for n sufficiently large, then $E(x)$ is homeomorphic to the Cantor set C.

III. If $|x(n)| \leqslant \sum_{i>n} |x(i)|$ for n sufficiently large, then $E(x)$ is a finite union of closed intervals. Moreover, if $|x(n)| \geqslant |x(n+1)|$ for almost all n, and $E(x)$ is a finite union of closed intervals, then $|x(n)| \leqslant \sum_{i>n} |x(i)|$ for n sufficiently large.

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1980 - The Weinstein-Shapiro sequence

1984 - The Ferens sequence

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Theorem Guthrie-Nymann

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For any $x \in l_1 \setminus c_{00}$, the set $E(x)$ is one of the following types:

$(\mathcal{I})$ *a finite union of closed intervals;*

$(\mathcal{C})$ *homeomorphic to the Cantor set;*

$(\mathcal{MC})$ *homeomorphic to the set $E(c)$ (of subsums of the sequence $(\frac{3}{4}, \frac{2}{4}, \frac{3}{16}, \frac{2}{16}, \frac{3}{64}, \dots)$).*

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$$l_1 = c_{00} \cup \mathcal{C} \cup \mathcal{I} \cup \mathcal{MC}$$

- Some algebraic and topological properties of these sets have been recently considered in - [B,B,G,S]

- The structure of the achievement sets $E(x)$ for multigeometric sequences x was studied in - [B,F,S]

where by *multigeometric sequence* we understand a sequence of the form

$$(k_0, k_1, \ldots, k_m, k_0 q, k_1 q, \ldots, k_m q, k_0 q^2, k_1 q^2, \ldots, k_m q^2, k_0 q^3 \ldots)$$

for some positive numbers $k_0, \ldots, k_m$ and $q \in (0, 1)$

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Set $K(\Sigma; q)$

The achievement sets of multigeometric sequences are particular cases of self-similar sets of the form

$$K(\Sigma; q) = \left\{ \sum_{n=0}^{\infty} a_n q^n : (a_n)_{n=0}^{\infty} \in \Sigma^{\omega} \right\}$$

where $\Sigma \subset \mathbb{R}$ and $q \in (0,1)$.

The set $K(\Sigma; q)$ is self-similar in the sense that $K(\Sigma; q) = \Sigma + q \cdot K(\Sigma; q)$.

Moreover, the set $K(\Sigma; q)$ can be found as a unique compact solution $K \subset \mathbb{R}$ of the equation $K = \Sigma + qK$.

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It is easy to see that for a multigeometric sequence $x_q = (k_0, \ldots, k_m; q)$ the achievement set $E(x)$ coincides with the self-similar set $K(\Sigma; q)$ for the set

$$\Sigma = \left\{ \sum_{n=0}^{m} k_n \varepsilon_n : (\varepsilon_n)_{n=0}^{m} \in \{0,1\}^{m+1} \right\}$$

of all possible sums of the numbers $k_0, \ldots, k_m$.

Definitions

For a compact $A \subset \mathbb{R}$ let us denote:

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in particular for finite set $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ (where $\sigma_1 < \cdots < \sigma_s$) we have

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Let $\Sigma = \{\sigma_1, \dots, \sigma_s\}$ for some real numbers $\sigma_1 < \cdots < \sigma_s$. The self-similar sets $K(\Sigma; q)$ where $q \in (0,1)$ have the following properties:

1. *$K(\Sigma; q)$ is an interval if and only if $q \geqslant I(\Sigma)$;*
2. *$K(\Sigma; q)$ is not a finite union of intervals if $q < I(\Sigma)$ and $\Delta(\Sigma) \in \{\sigma_2 - \sigma_1, \sigma_s - \sigma_{s-1}\}$;*
3. *$K(\Sigma; q)$ contains an interval if $q \geqslant i(\Sigma)$;*
4. *If $d = \frac{\delta(\Sigma)}{\operatorname{diam}(\Sigma)} < \frac{1}{3+2\sqrt{2}}$ and $\frac{1}{|\Sigma|} < \frac{\sqrt{d}}{1+\sqrt{d}}$, then for almost all $q \in \left(\frac{1}{|\Sigma|}, \frac{\sqrt{d}}{1+\sqrt{d}}\right)$ the set $K(\Sigma; q)$ has positive Lebesgue measure and the set $K(\Sigma; \sqrt{q})$ contains an interval;*

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4. *If* $d = \frac{\delta(\Sigma)}{\mathrm{diam}(\Sigma)} < \frac{1}{3+2\sqrt{2}}$ *and* $\frac{1}{|\Sigma|} < \frac{\sqrt{d}}{1+\sqrt{d}}$*, then for almost all* $q \in \left(\frac{1}{|\Sigma|}, \frac{\sqrt{d}}{1+\sqrt{d}}\right)$ *the set* $K(\Sigma; q)$ *has positive Lebesgue measure and the set* $K(\Sigma; \sqrt{q})$ *contains an interval;*

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1. $K(\Sigma; q)$ is a Cantor set of zero Lebesgue measure if $q < \frac{1}{|\Sigma|}$ or, more generally, if $q^n < \frac{1}{|\Sigma_n|}$ for some $n \in \mathbb{N}$ where $\Sigma_n = \{\sum_{k=0}^{n-1} a_k q^k : (a_k)_{k=0}^{n-1} \in \Sigma^n\}$.
2. If $\Sigma \supset \{a, a+1, b+1, c+1, b+|\Sigma|, c+|\Sigma|\}$ for some real numbers $a, b, c \in \mathbb{R}$ with $b \neq c$, then there is a strictly decreasing sequence $(q_n)_{n\in\omega}$ with $\lim_{n\to\infty} q_n = \frac{1}{|\Sigma|}$ such that the sets $K(\Sigma; q_n)$ has Lebesgue mesure zero.

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Theorem

If there exists $n \in \mathbb{N}$ such that

$$\Big| \sum_{i=0}^{n-1} q^i \Sigma \Big| \cdot q^n < 1$$

then the set $K(\Sigma, q)$ has measure zero.

Proof

$K := K(\Sigma, q)$

$$K = \Sigma + qK$$

$$K = \sum_{i=0}^{n-1} q^i \Sigma + q^n K$$

$\Sigma_n := \sum_{i=0}^{n-1} q^i \Sigma$

$$|\Sigma_n| \cdot q^n < 1$$

$$\lambda(K) > 0$$

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Lemma

For any integer numbers $s > 1$ and $n > 1$ the unique positive solution q of the equation

$$x + x^2 + ... + x^{n-1} = \frac{1}{s-1} \tag{1}$$

is greater then $\frac{1}{s}$. Moreover, there is $n_0 \in \mathbb{N}$ such that for any $n > n_0$

$$\left(s^n - 2^{n-1}\right) \cdot q^n < 1. \tag{2}$$

Proof

$$\sum_{i=1}^{n-1}\left(\frac{1}{s}\right)^i = \frac{1}{s-1}\cdot\left(1-\frac{1}{s^{n-1}}\right) < \frac{1}{s-1},$$

$$q > \frac{1}{s}$$

$$\frac{1}{s-1} = \sum_{i=1}^{n-2}\left(\frac{1}{s}\right)^i + \frac{1}{(s-1)\,s^{n-2}}$$

$$q^{n-1} = \frac{1}{s-1} - \sum_{i=1}^{n-2} q^i < \frac{1}{s-1} - \sum_{i=1}^{n-2}\left(\frac{1}{s}\right)^i = \frac{1}{(s-1)\,s^{n-2}}.$$

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Theorem

If a finite subset $\Sigma \subset \mathbb{R}$ contains the set $\{a, a+1, b+1, c+1, b+|\Sigma|, c+|\Sigma|\}$ for some real numbers a, b, c with $b \neq c$, then there is a decreasing sequence $(q_n)_{n=1}^{\infty}$ tending to $\frac{1}{|\Sigma|}$ such that, for any $n \in \mathbb{N}$, the self-similar set $K(\Sigma, q_n)$ has Lebesgue measure zero.

Proof

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Let $s = |\Sigma|$ and for every n denote by q_n the unique positive solution of the equation (1) from Lemma. Let n_0 be a natural number such that

$$\left(s^n - 2^{n-1}\right) \cdot (q_n)^n < 1$$

for any $n > n_0$. Clearly $(q_n)_{n=n_0}^{\infty}$ is a decreasing sequence and $\lim_{n\to\infty} q_n = \frac{1}{s}$. It suffices to show that $K(\Sigma, q)$ has measure zero for $n > n_0$.

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Proof

Taking into account that each q_n is a solution of (1), we conclude that

$$a+\sum_{i=1}^{n-1}(s-1+\varepsilon_i)(q_n)^i=(a+1)+\sum_{i=1}^{n-1}\varepsilon_i(q_n)^i$$

for any $\varepsilon_i\in\{b+1,c+1\}\subset\Sigma$. Therefore

$$\left|\sum_{i=1}^{n-1}(q_n)^i\,\Sigma\right|\leqslant s^n-2^{n-1}.$$

Hence, by Lemma,

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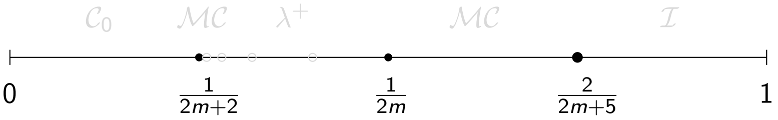
Hence, by Lemma,

$$\left| \sum_{i=1}^{n-1} (q_n)^i \, \Sigma \right| \cdot (q_n)^n < 1.$$

and we can apply Theorem to conclude that $K(\Sigma, q)$ has Lebesgue measure zero.

For sequence $x = (3, \overbrace{2, \dots, 2}^{m-times}; q)$ we know, that



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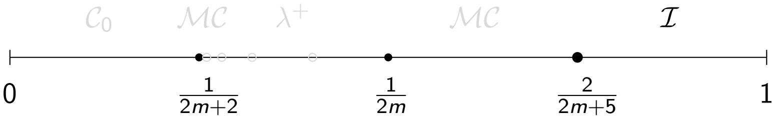

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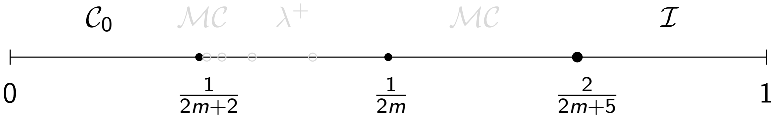

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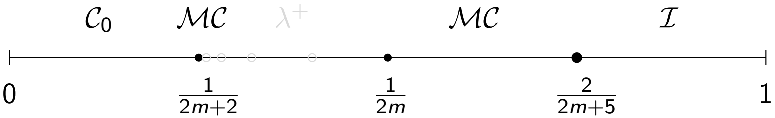

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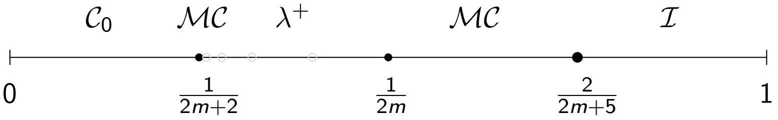

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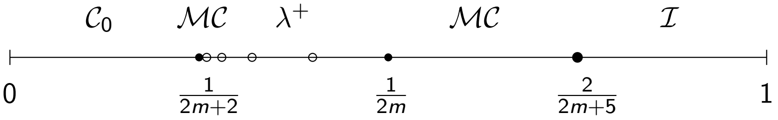

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