A construction of commutative rings with unit on $\mathbb{R}^4$ with coordinate-wise addition

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The 28th International Summer Conference
on Real Functions Theory
Stará Lesná, August 31 - September 5, 2014

Motivation for multi-polarity

physics:

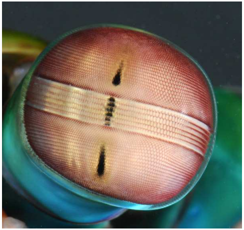
- K-phase electric current
  - engines with $K = 3, 4, 6, 12, \ldots$
- multi-polar magnets, telescopes, microscopes

biology:

- K-receptor color perception system
  - $K = 1$ … sea mammals
  - $K = 2$ … most mammals
  - $K = 3$ … most people and some apes
  - $K = 4$ … some women, most birds, some insects
  - $K = 5$ … some birds (pigeons), some butterflies
  - $K = 16$ … mantis shrimp

Motivation for multi-polarity

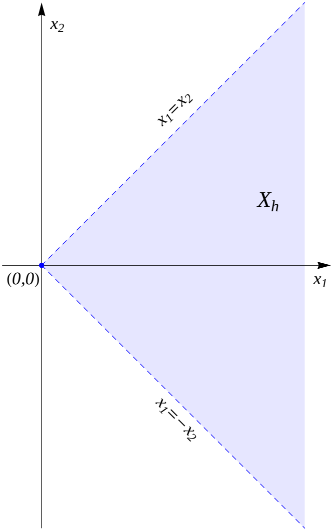
Mantis shrimp



- multi-polarity is a generalization of the notion of vector space
- every vector space is equipped with two polarities "$+$" and "$-$"
- a semi-field is used instead of a field
- semi-field – field without the requirement of existence of additive inverses
- "non-polar" objects (neutral elements) – elements of a semi-field
- "polarizing" objects (generalized signs) – algebra of some operators defined on the semi-field

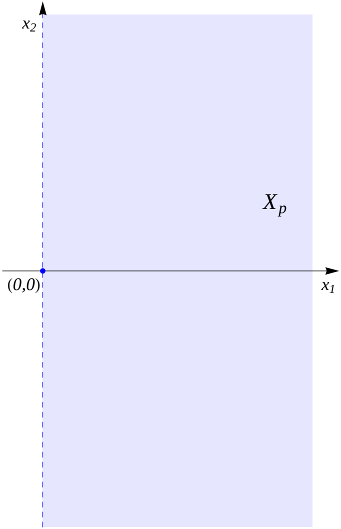
Semi-field of double numbers

- $(a, b) + (c, d) = (a + c, b + d)$
- $(a, b) \cdot (c, d) = (ac + bd, ad + bc)$
- double numbers = split-complex numbers = semi-complex numbers = hyperbolic complex numbers = hyperbolic numbers
- $X_h = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > |x_2|\} \cup \{0, 0\}$
- $(X_h, +, \cdot)$ is a semi-field of double numbers



Semi-field of dual numbers

- $(a,b)+(c,d)=(a+c,b+d)$
- $(a,b)\cdot(c,d)=(ac,ad+bc)$
- dual numbers = parabolic complex numbers = parabolic numbers
- $X_p=\{(x_1,x_2)\in\mathbb{R}^2 \mid x_1>0\}\cup\{0,0\}$
- $(X_p,+,\cdot)$ is a semi-field of dual numbers



Algebra of operators

- let us consider three operators

$$\begin{array}{rcl} A & = & (a_1, a_2, a_3, a_4) \\ B & = & (b_1, b_2, b_3, b_4) \\ C & = & (c_1, c_2, c_3, c_4) \end{array}$$

  $a_i, b_i, c_i \in \mathbb{R}, i = 1, 2, 3, 4$

- $D : X \to \mathbb{R}^4$

$$D(x_1, x_2) = \\ (d_1x_1 - d_2x_2, d_1x_2 + d_2x_1, d_3x_1 + d_4x_2, -d_3x_2 + d_4x_1)$$

  for $D \in \{A, B, C\}$

Algebra of operators

$\otimes$	A	B	C
A	A	B	C
B	B	C	A
C	C	A	B

- $$\begin{aligned} A &= (1,0,0,0) \\ B_{r,s,t} &= \left(-\frac{1}{2}, \frac{r}{2}\sqrt{3+4s^2+4t^2}, s, t\right) \\ C_{r,s,t} &= \left(-\frac{1}{2}, -\frac{r}{2}\sqrt{3+4s^2+4t^2}, -s, -t\right), \end{aligned}$$

  $r \in \{-1,1\}, s \in \mathbb{R}, t \in \mathbb{R}$

- $m = (x, y, z) \in X^3$, formally $m = Ax + By + Cz$
- *cancellation law*:

$$Au + B_{r,s,t}u + C_{r,s,t}u = 0$$

  for all $u \in X$
- equivalence relation $\cong$: for all $(x, y, z) \in X^3, u \in X$

$$(x, y, z) \cong (x + u, y + u, z + u)$$

Operations in the three-polar space

- addition:

$$(x, y, z) \oplus (u, v, w) = (x + u, y + v, z + w)$$

- formally

$$\begin{aligned} -Au &= Bu + Cu, \\ -Bv &= Av + Cv, \\ -Cw &= Aw + Bw, \end{aligned}$$

subtraction:

$$(x, y, z) \ominus (u, v, w) = (x + v + w, y + u + w, z + u + v)$$

Operations in the three-polar space

- multiplication:

$$(x, y, z) \odot (u, v, w) = (Ax + By + Cz) \odot (Au + Bv + Cw)$$

$$= (xu + yw + zv, xv + yu + zw, xw + yv + zu)$$

$$Cx \odot By = (C \otimes B)(x \cdot y) = Axy$$

- conjugation:

$$(u, v, w)^* = (u, w, v)$$

$$(u, v, w) \odot (u, v, w)^* \cong (x, 0, 0) \text{ for some } x \in X$$

- division:

$$(x, y, z) \oslash (u, v, w) = \frac{(x, y, z) \odot (u, v, w)^*}{(u, v, w) \odot (u, v, w)^*},$$

if $(u, v, w) \odot (u, v, w)^*$ is invertible in X

- $\mathbb{M} := X^3 \backslash_{\cong}$
- let be $\mathcal{R} : \mathbb{M} \to \mathbb{R}^4$

$$\mathcal{R}(x, y, z) = Ax + B_{r,s,t}y + C_{r,s,t}z$$

  if $s^2 + t^2 > 0$, then $\mathcal{M} = \mathcal{R}^{-1} : \mathbb{R}^4 \to \mathbb{M}$
- both mappings $\mathcal{R}, \mathcal{M}$ are bijective and linear

Operations on $\mathbb{R}^4$

$\mathbf{p}, \mathbf{q} \in \mathbb{R}^4$,

$$\mathbf{p} \boxplus \mathbf{q} := \mathcal{R}(\mathcal{M}(\mathbf{p}) \oplus \mathcal{M}(\mathbf{q})) = \mathcal{R}(\mathcal{M}(\mathbf{p})) + \mathcal{R}(\mathcal{M}(\mathbf{q})) = \mathbf{p} + \mathbf{q}$$

$$\mathbf{p} \boxdot \mathbf{q} := \mathcal{R}(\mathcal{M}(\mathbf{p}) \odot \mathcal{M}(\mathbf{q}))$$

let be $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ a basis in $\mathbb{R}^4$ and

$$\mathbf{p} = \sum_{i=1}^{4} p_i \mathbf{e}_i, \quad \mathbf{q} = \sum_{j=1}^{4} q_j \mathbf{e}_j,$$

then

$$\mathbf{p} \boxdot \mathbf{q} = \sum_{i,j=1}^{4} p_i q_j \, \mathcal{R}(\mathcal{M}(\mathbf{e}_i) \odot \mathcal{M}(\mathbf{e}_j))$$

- $(\mathbb{R}^4, \boxplus, \boxdot)$ is a commutative ring with unit

$r = 1, s = 1, t = 0$

$$A = (1,0,0,0),\ B = \left(-\frac{1}{2}, \frac{\sqrt{7}}{2}, 1, 0\right),\ C = \left(-\frac{1}{2}, -\frac{\sqrt{7}}{2}, -1, 0\right)$$

for $\mathbf{p}, \mathbf{q} \in \mathbb{R}^4$

$$\begin{aligned} \mathbf{p} \boxdot \mathbf{q} \quad = \quad & \Big(p_1q_1 + p_2q_2 + \frac{9}{2}p_3q_3 - \frac{5}{2}p_4q_4 - \sqrt{7}(p_3q_2 + p_2q_3), \\ & p_1q_2 + p_2q_1 - \sqrt{7}(p_2q_4 + p_4q_2) + \frac{5}{2}(p_3q_4 + p_4q_3), \\ & p_1q_3 + p_3q_1 - p_2q_4 - p_4q_2, \\ & p_1q_4 + p_4q_1 - p_2q_3 - p_3q_2 + \sqrt{7}(p_3q_3 - p_4q_4)\Big). \end{aligned}$$

Example with the semi-field of dual numbers

$r = 1, s = 1, t = 0$

$$A = (1,0,0,0),\ \ B = \left(-\frac{1}{2}, \frac{\sqrt{7}}{2}, 1, 0\right),\ \ C = \left(-\frac{1}{2}, -\frac{\sqrt{7}}{2}, -1, 0\right)$$

for $\mathbf{p}, \mathbf{q} \in \mathbb{R}^4$

$$\begin{aligned}
\mathbf{p} \boxdot \mathbf{q} \quad = \quad & \Bigg(p_1 q_1 + \frac{11}{4} p_3 q_3 - \frac{7}{4} p_4 q_4 - \frac{\sqrt{7}}{2}(p_3 q_2 + p_2 q_3), \\
& p_1 q_2 + p_2 q_1 - \frac{\sqrt{7}}{2}(p_2 q_4 + p_4 q_2) + \frac{3}{4}(p_3 q_4 + p_4 q_3), \\
& p_1 q_3 + p_3 q_1 - \frac{\sqrt{7}}{2}(p_3 q_4 + p_4 q_3), \\
& p_1 q_4 + p_4 q_1 - p_2 q_3 - p_3 q_2 + \sqrt{7}(p_3 q_3 - p_4 q_4)\Bigg).
\end{aligned}$$

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