ALGEBRABILITY WITHIN THE CLASS OF BAIRE 1 FUNCTIONS

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Let κ be a cardinal. A subset A of a linear commutative algebra is κ-algebrable if $A \cup \{0\}$ contains a κ-generated algebra B, i.e. the minimal cardinality of system of generators of B is equal to κ.

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Let κ be a cardinal. A subset A of a linear commutative algebra is is strongly κ-algebrable if $A \cup \{0\}$ contains a κ-generated algebra B that is isomorphic with a free algebra.

Exponential-like function method

Definition

We say that a function $f : \mathbb{R} \to \mathbb{R}$ *is exponential-like of a range* m *if it is given by* $f(x) = \sum_{i=1}^{m} a_i e^{\beta_i x}$ *for some distinct nonzero numbers* $\beta_1, \ldots, \beta_m$ *and some nonzero real numbers* $a_1, \ldots, a_m$.

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Property

For every positive integer m *any exponential-like function* $f : \mathbb{R} \to \mathbb{R}$ *of range* m, *and each* $c \in \mathbb{R}$, *the preimage* $f^{-1}[\{c\}]$ *has at most* m *elements. Consequently,* f *is not constant in every subinterval of* $\mathbb{R}$.

Exponential-like function method

Theorem

Given a family $\mathcal{F} \subset \mathbb{R}^{[0,1]}$, assume that there exists a function $F \in \mathcal{F}$ such that $f \circ F \in \mathcal{F} \setminus \{0\}$ for every exponential-like function $f : \mathbb{R} \to \mathbb{R}$. Then $\mathcal{F}$ is strongly $\mathfrak{c}$-algebrable. More exactly, if $H \subset \mathbb{R}$ is a set of cardinality $\mathfrak{c}$, linearly independent over the rationals $\mathbb{Q}$, then $exp \circ (rF)$, $r \in H$, are free generators of an algebra contained in $\mathcal{F} \cup \{0\}$.

Known facts

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$$\mathcal{C}_{nat} \subsetneq \mathcal{C}_d \subsetneq \mathcal{DB}_1 \subsetneq \mathcal{B}_1$$

- the set $\mathcal{C}_d \setminus \mathcal{C}_{nat}$ is strongly $\mathfrak{c}$-algebrable;
- the set $\mathcal{DB}_1 \setminus \mathcal{C}_d$ is strongly $\mathfrak{c}$-algebrable;
- the set $\mathcal{B}_1 \setminus \mathcal{DB}_1$ is strongly $\mathfrak{c}$-algebrable.

Topologies generated by operators

Let consider operator $\Phi : \mathcal{L} \to 2^{\mathbb{R}}$ which fulfills the following conditions: for any measurable sets $A, B \subset \mathbb{R}$

1. $\Phi(\emptyset) = \emptyset, \quad \Phi(\mathbb{R}) = \mathbb{R}$;
2. $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$;
3. $A \triangle B \in \mathcal{I} \Rightarrow \Phi(A) = \Phi(B)$;
4. $\lambda(\Phi(A) \setminus A) = 0$.

$$\mathcal{T}_\Phi = \{A \in \mathcal{L} \colon A \subset \Phi(A)\}$$

Basic Theorem

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Let Φ_1 and Φ_2 be operators generating the topologies $\mathcal{T}_{\Phi_1}$ and $\mathcal{T}_{\Phi_2}$ invariant under translation and stronger than natural topology on $\mathbb{R}$. If there are the sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$, $(d_n)_{n\in\mathbb{N}}$ converging to zero such that:

(1) $0 < b_{n+1} < a_n < c_n < d_n < b_n$ *for any* $n \in \mathbb{N}$,

(2) *the interval sets* $A = \bigcup_{n=1}^{\infty} [a_n, b_n]$ *and* $C = \bigcup_{n=1}^{\infty} [c_n, d_n]$ *have the property: 0 is a* Φ_1*-dispersion point of* A *and is not a* Φ_2*-dispersion point of the set* C,

then the family $\mathcal{C}_{\Phi_1} \setminus \mathcal{C}_{\Phi_2}$ *is strongly* $\mathfrak{c}$*-algebrable.*

Algebrability - ψ-density case

Let $\widehat{\mathcal{C}}$ be be the family of nondecreasing continuous functions $\psi : (0, \infty) \to (0, \infty)$ such that $\lim_{x \to 0+} \psi(x) = 0$.

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Proposition

Let $\psi_1, \psi_2 \in \widehat{\mathcal{C}}$. If $\mathcal{T}_{\psi_1} \setminus \mathcal{T}_{\psi_2} \neq \emptyset$, then there exist interval sets A and C satisfying the conditions (1) and (2) from Basic Theorem.

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Corollary

If $\mathcal{T}_{\psi_1} \setminus \mathcal{T}_{\psi_2} \neq \emptyset$, then $\mathcal{C}_{\psi_1} \setminus \mathcal{C}_{\psi_2}$ is strongly $\mathfrak{c}$-algebrable.

Algebrability - $\langle s \rangle$-density case

Denote by $\mathcal{S}$ the family of nondecreasing and unbounded sequences of positive numbers. Let $\langle s \rangle = (s_n)_{n \in \mathbb{N}} \in \mathcal{S}$. We will say that $x \in \mathbb{R}$ is *a right-hand $\langle s \rangle$-density point* of a measurable set A if

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Corollary

If $\mathcal{T}_{\langle s \rangle} \setminus \mathcal{T}_{\langle t \rangle} \neq \emptyset$, then $\mathcal{C}_{\langle s \rangle} \setminus \mathcal{C}_{\langle t \rangle}$ is strongly $\mathfrak{c}$-algebrable.

The oscillation index classification of Baire 1 class

Let $f\colon E \to \mathbb{R}$, $E \subset \mathbb{R}$ and take $\varepsilon > 0$. For any $A \subset E$ let

$$P_\varepsilon(A) = \Big\{x \in A : \operatorname{osc}(f, x, A) \geq \varepsilon\Big\}.$$

Let us define the transfinite sequence of closed sets $(F_\varepsilon^\alpha)_{\alpha<\omega_1}$ in the following way

$$F_{f,\varepsilon}^\alpha = \begin{cases} E & \text{for } \alpha = 0 \\ P_\varepsilon(F_{f,\varepsilon}^\beta) & \text{for } \alpha = \beta + 1 \\ \bigcap_{\beta<\alpha} F_{f,\varepsilon}^\beta & \text{when } \alpha \text{ is limit ordinal.} \end{cases}$$

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Let

$$\beta(f,\varepsilon)=\begin{cases}\text{the smallest } \alpha \text{ such that } F^{\alpha}_{f,\varepsilon}=\emptyset, & \text{if such exists,}\\ \omega_1, & \text{otherwise.}\end{cases}$$

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The number $\beta(f) = \sup\limits_{\varepsilon>0} \beta(f,\varepsilon)$ is called the oscillation index of f.

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Theorem

Let $\alpha < \omega_1$ and $\mathcal{G}_\alpha = \{g \in \mathbb{R}^{\mathbb{R}} : \beta(g) = \alpha\}$. Then the set of functions from $\mathcal{G}_\alpha$ which are not Darboux is strongly $\mathfrak{c}$-algebrable.

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