Real functions in generalized probability

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Abstract

Measurable fuzzy sets, i. e. [0,1]-valued measurable functions, equipped with suitable algebraic structures and a convergence of sequences, model fuzzy random events. We give a classification scheme for classes of [0,1]-valued functions related to various properties of the model of fuzzy probability and discuss some applications.

Why real functions in probability theory?

C(lassical) P(robability) T(heory)

- random events: measurable sets and set (logical) operations - can be represented via indicator functions and the corresponding (Boolean sum and product, negation) operations on functions;
- a (reduced) σ-field $\mathbb{A}$ of subsets of a set X, $A \in \mathbb{A}$, $\chi_A \subseteq \{0,1\}^X, \chi_A(x) = 1$ for $x \in A$ and $\chi_A(x) = 0$ otherwise;
- (countable) operations: union (or), intersection (and), complement (negation);
- probability measure: a normed σ-additive function $P : \mathbb{A} \longrightarrow [0,1]$;
- random variable: a measurable function $f : X \longrightarrow R$

Why real functions in probability theory?

F(uzzy) P(robability) T(heory)

- fuzzy random events: the structured set $\mathcal{M}(\mathbb{A})$ of all measurable functions ranging in $[0,1]$ (suitable MV-algebras);
- operations on fuzzy random events: Łukasiewicz operations;
- $\mathbb{A}$ and $\mathcal{M}(\mathbb{A})$ ... Łukasiewicz tribes, D-posets of fuzzy sets
- D-poset ... $(X, \leq, 0, 1, \ominus)$
- fuzzy observable ... sequentially continuous D-homomorphism of $\mathcal{M}(\mathbb{A})$ into $\mathcal{M}(\mathbb{B})$
- state ... $\int (.) dp$, p is a probability measure on $\mathbb{A}$, $(.) \in \mathcal{M}(\mathbb{A})$ (states: exactly sequentially continuous D-homomorphisms)

  REMARK. For $\mathbb{T} = \{\emptyset, \{a\}\}$, $\mathcal{M}(\mathbb{T}) \equiv [0,1]$, hence a state is a fuzzy observable into $[0,1]$.

Basic notions

Fuzzy set ... a [0,1]-valued function on a set X, $u \in [0,1]^X$

D-poset of fuzzy sets ... $\mathcal{X} \subseteq [0,1]^X$ such that

- pointwise partial order;
- $0_X, 1_X \in \mathcal{X}$;
- if $u, v \in \mathcal{X}$ and $v \leq u$, then $u - v \in \mathcal{X}$ (pointwise);
- partial operation $\ominus$: $u \ominus v$ is defined as $u - v$ iff $v \leq u$;
- introduced by F. Kôpka and F. Chovanec

Bold algebra ... a system $\mathcal{X} \subseteq [0,1]^X$ of fuzzy sets equipped with Łukasiewicz operations

$(u \oplus v)(x) = min\{u(x) + v(x), 1\}$, $x \in X$,
$(u \odot v)(x) = max\{u(x) + v(x) - 1, 0\}$, $x \in X$,
$u^c(x) = 1 - u(x)$, $x \in X$.

REMARK. Bold algebras generalize fields of subsets:
$\chi_{A \cup B}(x) = min\{\chi_A(x) + \chi_B(x), 1\}$, $x \in X$;
$\chi_{A \cap B}(x) = max\{\chi_A(x) + \chi_B(x) - 1, 0\}$, $x \in X$;
$\chi^c_{X \setminus A}(x) = 1 - \chi_A(x)$, $x \in X$.

Recently, B. Riečan has developed IF-probability based on special fuzzy sets, cf.

B. Riečan: Analysis of fuzzy logic models. In: Intelligent Systems (ed. V. M. Koleshko), In Tech, Rijeka 2012, 219–244.

Let X be a set. An *IF*-subset of X is a pair $A = (\mu_A, \nu_A)$, where μ_A, ν_A are fuzzy subsets of X (called the membership and nonmembership functions of A, respectively) and $\mu_A + \nu_A \leq 1_X$. Clearly, for $\nu_A = 1_X - \mu_A$, (μ_A, ν_A) can be considered as a fuzzy subset of X. Measurable *IF*-subsets form generalized random events in the *IF*-probability.

Pairs of fuzzy sets lead to products of D-posets and special models within the fuzzy probability theory.
Let $\mathcal{X} \subseteq I^X$ and $\mathcal{Y} \subseteq I^Y$ be D-posets of fuzzy sets. Let $\mathcal{Z} \subseteq I^Z$ be their product. Then $\mathcal{Z}$ consists of all pairs $(u, v), u \in \mathcal{X}, v \in \mathcal{Y}$, where the partial order, difference, convergence is defined coordinatewise, and Z is the disjoint union of X and Y (their coproduct in the category of sets and maps). Each $w = (u, v)$ can be visualized as a function on Z, where u and v are "disjoinly glued" to form w.

If $\mathcal{X}$ is a bold algebra, then $\mathcal{X} \times \mathcal{X}$ denotes the corresponding power bold algebra. In particular, if $\mathcal{X} = \mathcal{M}(\mathbb{A})$, then $\mathcal{M}(\mathbb{A}) \times \mathcal{M}(\mathbb{A})$ carries the usual coordinatevise Łukasiewicz operations: $\oplus, \odot$ and complementation.

Recall that a D-poset is a partially ordered set equipped with a partial operation "difference". D-poset structure preserving maps as morphisms play a fundamental role in generalized probability, e.g., classical probability measures are exactly sequentially continuous D-homomorphisms on a σ-field into [0,1].
On the other hand, bold algebras (modeling fuzzy random events) are in terms of "addition" and "complementation". We introduce a new structure based on "addition" which leads to a better understanding of the transition from D-posets to bold algebras and the transition from the CPT to the FPT.
REMERK. Each bold algebra can be reorganized into a D-poset (indirect).

Bold algebras and l-groups

Let X be a set and let $G \subseteq X^R$ be an l-group (lattice group) of bounded real functions on X such that $1_X \in G$. Then

$\mathcal{X}_G = \{u \in G;\, 0_X \leq u \leq 1_X\}$ equipped with the Łukasiewicz operations
$(u \oplus v)(x) = min\{u(x) + v(x), 1\}$, $x \in X$,
$(u \odot v)(x) = max\{u(x) + v(x) - 1, 0\}$, $x \in X$,
$u^c(x) = 1 - u(x)$, $x \in X$,
is a bold algebra and all bold algebras are of this type.

REMARK. Like in fields of sets, the operation $\odot$ can be redefined in terms of $\oplus$ and $(.)^c$: $u \odot v = (u^c \oplus v^c)^c$.

A-posets

An A-**poset** is a system $\mathbf{S} = (S, \leq, 0, 1, \oplus)$ consisting of a partially ordered set S with the least element 0, the greatest element 1 and a partial binary operation $\oplus$ satisfying the following conditions:

(A1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$;

(A2) If $(a \oplus b) \oplus c$ is defined, then $a \oplus (b \oplus c)$ is defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;

(A3) For each $a \in S$ there exists a unique $b \in S$ such that $a \oplus b = 1$;

(A4) If $a \oplus b$ is defined, $a_1 \leq a$ and $b_1 \leq b$, then $a_1 \oplus b_1$ is defined and $a_1 \oplus b_1 \leq a \oplus b$.

The element $b \in S$ from axiom (A3) is called the **complement** of a and it is denoted as a^c.

Remarks

- $a \oplus 0 = a$, for all $a \in S$
- $(A4)$ is equivalent to ($a \oplus b$ is defined iff $a \leq b^c$)
- A-posets and D-posets are equivalent

GENERALIZATIONS

- in $(A3)$, the complement need not be unique
- leave out the top element and $(A3)$
- simplex-type probability domains

Classification scheme

- A-posets;
- IA-posets ... A-posets with enough states;
- bold algebras ... lattice IA-posets;
- Łukasiewicz tribes ... sequentially closed lattice IA-posets;
- generated Łukasiewicz tribes ($\mathcal{M}(\mathbb{A})$) ... divisible detto.

Convex combinations

Let $\mathcal{X}_1$, $\mathcal{X}_2$, ..., $\mathcal{X}_n$, $n > 1$, be *ID*-posets and let $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times ... \times \mathcal{X}_n$ be their product. Let $h_i : \mathcal{X}_i \longrightarrow I$, $i = 1, 2, ..., n$, be sequentially continuous D-homomorphisms and let $\alpha_i \in I$, $i = 1, 2, ..., n$, $\sum_{i=1}^{n} \alpha_i = 1$. For $u = (u_1, u_2, ..., u_n) \in \mathcal{X}$, put $h(u) = \sum_{i=1}^{n} \alpha_i h_i(u_i)$ and denote $h = \sum_{i=1}^{n} \alpha_i h_i$ the resulting map. Then h is said to be a convex combination of $h_1, h_2, ..., h_n$.

Lemma

h is a sequentially continuous D-homomorphism.