**Faculty of Mathematics
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On semiregularization of some abstract density topologies

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Let $\langle X, \tau \rangle$ be a topological Baire space and $\mathbb{K}(\tau)$, $\mathcal{B}a(\tau)$ denote the family of all meager sets and the family of all sets having the Baire property in $\langle X, \tau \rangle$, respectively. By $\mathcal{B}(\tau)$ we shall denote the family of Borel sets in $\langle X, \tau \rangle$.

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Let τ_0 stands for the natural topology on $\mathbb{R}$ and the family of meager sets, having the Baire property and Borel sets in $\langle \mathbb{R}, \tau_0 \rangle$ is denoted by $\mathbb{K}$, $\mathcal{B}a$ and $\mathcal{B}$, respectively.
By $\mathcal{C}(\langle X, \tau \rangle)$ we shall denote the family of all continuous transformations from $\langle X, \tau \rangle$ to $\langle \mathbb{R}, \tau_0 \rangle$.

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Let λ be the Lebesgue measure on $\mathbb{R}$, $\mathcal{L}$ the family of Lebesgue measurable sets and $\mathbb{L}$ the family of null Lebesgue sets.

If $\mathcal{C}$, $\mathcal{D}$ are families of subsets of X, then

$$\mathcal{C} \ominus \mathcal{D} = \{A \subset X \colon A = C \setminus D, C \in \mathcal{C}, D \in \mathcal{D}\}.$$

Let $\Phi\colon \tau \to 2^X$ be an operator satisfying the following conditions:

i) $\Phi(\emptyset) = \emptyset, \quad \Phi(X) = X$;

ii) $\underset{A,B\in\tau}{\forall} \Phi(A \cap B) = \Phi(A) \cap \Phi(B)$;

iii) $\underset{A\in\tau}{\forall} A \subset \Phi(A)$.

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Let $\mathbf{\Phi}(X, \tau)$ stand for the family of all operators satisfying the above conditions. We call this set as the family of all admissible operators on τ. It is well known that every set $A \in \mathcal{B}a(\tau)$ has the unique representation:

$$A = G(A) \vartriangle B,$$

where $G(A)$ is regular open set in $\langle X, \tau\rangle$ and $B \in \mathbb{K}(\tau)$.

Let $\Phi \in \mathbf{\Phi}(X, \tau)$. Let us define operator $\Phi_r \colon \mathcal{B}a(\tau) \to 2^X$ in the following way

$$\underset{A \in \mathcal{B}a(\tau)}{\forall} \Phi_r(A) = \Phi(G(A)).$$

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Theorem 1.

For every $\Phi \in \mathbf{\Phi}(X,\tau)$ operator Φ_r is a lower density operator on $(X, \mathcal{B}a(\tau), \mathbb{K}(\tau))$ and the family

$$\mathcal{T}_{\Phi_r} = \{A \in \mathcal{B}a(\tau) \colon A \subset \Phi_r(A)\}$$

is an abstract density topology on $(X, \mathcal{B}a(\tau), \mathbb{K}(\tau))$ containing τ and

a) *$A \in \mathbb{K}(\tau)$ if and only if A is a $\mathcal{T}_{\Phi_r}$-nowhere dense and $\mathcal{T}_{\Phi_r}$-closed;*

b) *$\mathbb{K}(\mathcal{T}_{\Phi_r}) = \mathbb{K}(\tau)$;*

c) *$\mathcal{B}a(\mathcal{T}_{\Phi_r}) = \mathcal{B}(\mathcal{T}_{\Phi_r}) = \mathcal{B}a(\tau)$;*

d) *$\langle X, \mathcal{T}_{\Phi_r} \rangle$ is a Baire space.*

Remark 1.

If $\langle X, \tau \rangle$ *is a topological discrete space, then for every operator* $\Phi \in \mathbf{\Phi}(X, \tau)$ *we get that* $\mathcal{T}_{\Phi_r} = \tau = 2^X$*.*

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If $\langle X, \tau \rangle$ contains a dense subset $A \in \mathbb{K}(\tau)$, then $X \setminus A \in \mathcal{T}_{\Phi_r} \setminus \tau$, so $\mathcal{T}_{\Phi_r} \setminus \tau \neq \emptyset$.

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Remark 3.

If Φ *is a lower (almost lower) density operator on* $(X, \mathcal{B}a(\tau), \mathbb{K}(\tau))$, *then* $\Phi \in \mathbf{\Phi}(X, \tau)$ *and* $\mathcal{T}_{\Phi_r} = \mathcal{T}_{\Phi}$, *where* $\mathcal{T}_{\Phi} = \{A \in \mathcal{B}a(\tau) \colon A \subset \Phi(A)\}$.

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Remark 4.

If Φ *is a lower (almost lower) density operator on* $(\mathbb{R}, \mathcal{L}, \mathbb{L})$, *then* $\Phi \in \mathbf{\Phi}(\mathbb{R}, \tau_0)$ *and the topologies* $\mathcal{T}_{\Phi_r}$ *and* $\mathcal{T}_{\Phi}$, *where* $\mathcal{T}_{\Phi}$ *is topology generated by operator* Φ, *are not comparable.*

Theorem 2.

Let $\langle X, \tau \rangle$ be a topological Baire space and $\Phi(A) = A$ for every $A \in \tau$. Then $\Phi \in \mathbf{\Phi}(X, \tau)$ and $\mathcal{T}_{\Phi_r} = \tau \ominus \mathbb{K}(\tau)$.

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Corollary 1.

$$\bigcap_{\Phi \in \mathbf{\Phi}(X,\tau)} \mathcal{T}_{\Phi_r} = \tau \ominus \mathbb{K}(\tau).$$

Let $\Phi \in \mathbf{\Phi}(X, \tau)$ and let

$$\mathcal{T}_{\Phi_r'} = \{A \subset X \colon A = W \cup B, W \in \tau, B \subset \Phi(W)\}$$
$$\mathcal{T}_{\Phi_r''} = \{A \subset X \colon A = W \cup B, W \in \tau, B \subset \Phi(G(W))\}.$$

Let $\Phi \in \mathbf{\Phi}(X, \tau)$ and let

$$\mathcal{T}_{\Phi'_r} = \{A \subset X \colon A = W \cup B, W \in \tau, B \subset \Phi(W)\}$$
$$\mathcal{T}_{\Phi''_r} = \{A \subset X \colon A = W \cup B, W \in \tau, B \subset \Phi(G(W))\}.$$

Theorem 3.

The families $\mathcal{T}_{\Phi'_r}$, $\mathcal{T}_{\Phi''_r}$ are topologies such that $\mathcal{T}_{\Phi'_r} \subset \mathcal{T}_{\Phi''_r} \subset \mathcal{T}_{\Phi_r}$. Moreover,

$$\mathcal{T}_{\Phi''_r} = \{A \in \mathcal{T}_{\Phi_r} \colon A = W \cup B, W \in \tau, B \in \mathbb{K}(\tau)\}.$$

Theorem 4.

If $\Phi \in \mathbf{\Phi}(X, \tau)$, *then*

$$\mathcal{T}_{\Phi'_r} \ominus \mathbb{K}(\tau) = \mathcal{T}_{\Phi''_r} \ominus \mathbb{K}(\tau) = \mathcal{T}_{\Phi_r}.$$

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This theorem implies that

Theorem 5.

If $\Phi \in \mathbf{\Phi}(X,\tau)$, *then*

$$\mathcal{C}(\langle X, \mathcal{T}_{\Phi'_r}\rangle) = \mathcal{C}(\langle X, \mathcal{T}_{\Phi''_r}\rangle) = \mathcal{C}(\langle X, \mathcal{T}_{\Phi_r}\rangle).$$

In many cases the topological space $\langle X, \mathcal{T}_{\Phi_r} \rangle$ is not completely regular and even not regular. Namely we have

Theorem 6.

If $\langle X, \tau \rangle$ is a topological Baire space such that there exists a τ-dense set $A \in \mathbb{K}(\tau)$, then for every $\Phi \in \mathbf{\Phi}(X, \tau)$ topological space $\langle X, \mathcal{T}_{\Phi_r} \rangle$ is not regular.

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If $\langle X, \tau \rangle$ is a topological Baire space such that there exists a τ-dense set $A \in \mathbb{K}(\tau)$, then for every $\Phi \in \mathbf{\Phi}(X, \tau)$ topological space $\langle X, \mathcal{T}_{\Phi_r} \rangle$ is not regular.

Since that we are looking for a coarsest topology $\mathcal{T} \subset \mathcal{T}_{\Phi_r}$ such that

$$\mathcal{C}(\langle X, \mathcal{T}_{\Phi_r} \rangle) = \mathcal{C}(\langle X, \mathcal{T} \rangle).$$

In some cases topology $\mathcal{T}_{\Phi'_r}$ satisfies this condition.

Example 1.

R. O'Malley introduced a.e. topology on $\mathbb{R}$*. A set* A *is a.e. open if* $A \in \mathcal{T}_d$ *and* $\lambda(A \setminus \operatorname{int}(A)) = 0$*.*
Let $\mathcal{T}_{a.e.}$ *be the family of all a.e. open sets.*

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If $\Phi = \Phi_d$, then $\Phi \in \mathbf{\Phi}(\mathbb{R}, \tau_0)$ and $\mathcal{T}_{a.e.} = \mathcal{T}_{\Phi'_r}$.

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O'Malley proved that the space $\langle \mathbb{R}, \mathcal{T}_{a.e.} \rangle$ is completely regular. Thus $\mathcal{T}_{\Phi'_r}$ is the coarsest topology such that

$$\mathcal{C}(\langle \mathbb{R}, \mathcal{T}_{\Phi_r} \rangle) = \mathcal{C}(\langle \mathbb{R}, \mathcal{T}_{\Phi'_r} \rangle).$$

In the next part we shall considered the topology included in $\mathcal{T}_{\Phi'_r}$.
Let $\langle X, \tau \rangle$ be a topological Baire space and $\Phi \in \mathbf{\Phi}(X, \tau)$.
We shall say that topology generated by $\mathcal{T}_{\Phi_r}$-regular open sets is **semiregularization** of topological space $\langle X, \mathcal{T}_{\Phi_r} \rangle$ and denoted by $\mathcal{T}_{\Phi_{ro}}$.
(A is $\mathcal{T}_{\Phi_r}$-regular open set if and only if $A = \Phi_r(A)$).
The concept of semiregularization of the abstract density topology was discussed by K. Ciesielski, W. Wilczyński, W. Wojdowski.

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Theorem 7.

If $\Phi \in \mathbf{\Phi}(X, \tau)$, then $\mathcal{T}_{\Phi_{ro}} \subset \mathcal{T}_{\Phi'_r}$ and $\mathcal{T}_{\Phi_{ro}} \ominus \mathbb{K}(\tau) = \mathcal{T}_{\Phi_r}$.

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$\mathcal{T}_{\Phi_{ro}} \subset \mathcal{T}_{\Phi'_r} \subset \mathcal{T}_{\Phi''_r} \subset \mathcal{T}_{\Phi_r}$ *and*
$\mathcal{C}(\langle X, \mathcal{T}_{\Phi_{ro}} \rangle) = \mathcal{C}(\langle X, \mathcal{T}_{\Phi'_r} \rangle) = \mathcal{C}(\langle X, \mathcal{T}_{\Phi''_r} \rangle) = \mathcal{C}(\langle X, \mathcal{T}_{\Phi_r} \rangle)$.

Evidently, in the case of operator Φ_d and O'Malley topology $\mathcal{T}_{a.e.}$ we have that $\mathcal{T}_{\Phi_{ro}} = \mathcal{T}_{a.e.}$.

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Theorem 7.

If $\Phi \in \mathbf{\Phi}(X, \tau)$, then $\mathcal{T}_{\Phi_{ro}} \subset \mathcal{T}_{\Phi'_r}$ and $\mathcal{T}_{\Phi_{ro}} \ominus \mathbb{K}(\tau) = \mathcal{T}_{\Phi_r}$.

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$\mathcal{T}_{\Phi_{ro}} \subset \mathcal{T}_{\Phi'_r} \subset \mathcal{T}_{\Phi''_r} \subset \mathcal{T}_{\Phi_r}$ *and*
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Evidently, in the case of operator Φ_d and O'Malley topology $\mathcal{T}_{a.e.}$ we have that $\mathcal{T}_{\Phi_{ro}} = \mathcal{T}_{a.e.}$.
The following example shows that semiregularization $\mathcal{T}_{\Phi_{ro}}$ does not have to be the coarsest topology such that

Example 2.

Let $A = (1, \infty)$, $B = (-\infty, -1)$. *The family* $\tau = \{\emptyset, \mathbb{R}, A, B, A \cup B\}$ *is a topology such that* $\mathcal{C}(\langle X, \tau \rangle) = const.$

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Let $\tau' = \tau \ominus J_\omega = \{A \subset \mathbb{R} \colon A = V \setminus P, V \in \tau, P \text{ is countable}\}$.
Then τ' *is the topological Baire space* T_1.

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Then τ' *is the topological Baire space* T_1.
Putting $\Phi(W) = W$ *for* $W \in \tau'$ *we get that* $\Phi \in \mathbf{\Phi}(\mathbb{R}, \tau')$ *and* $\mathcal{T}_{\Phi_r} = \tau' \ominus \mathbb{K}(\tau')$. *Hence*

$$\mathcal{C}(\langle X, \mathcal{T}_{\Phi_r} \rangle) = \mathcal{C}(\langle X, \tau' \rangle) = \mathcal{C}(\langle X, \tau \rangle) = const.$$

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Simultaneously $A \in \mathcal{T}_{\Phi_{ro}}$*, because* A *is* τ'*-regular open and*

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Simultaneously $A \in \mathcal{T}_{\Phi_{ro}}$, *because* A *is* τ'*-regular open and*

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Similarly $B \in \mathcal{T}_{\Phi_{ro}}$. *So that* $\mathcal{T}_{\Phi_{ro}}$ *is not the coarsest topology such that*

$$\mathcal{C}(\langle X, \mathcal{T}_{\Phi_{ro}} \rangle) = \mathcal{C}(\langle X, \mathcal{T}_{\Phi_r} \rangle).$$

because this topology is $\mathcal{T} = \{\emptyset, \mathbb{R}\}$.

The next theorem shows that under additional assumptions semiregularization $\mathcal{T}_{\Phi_{ro}}$ topological space $\langle \mathbb{R}, \mathcal{T}_{\Phi_r} \rangle$ is the coarsest topology such that

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Theorem 8.

Let $\Phi \in \mathbf{\Phi}(\mathbb{R}, \tau_o)$. If for every τ_0-regular open set V and for every $x \in \Phi(V)$ there exists at x an interval closed set $F \subset V$ such that $x \in \Phi(\operatorname{int}(F))$ and $\tau_0 \subset \mathcal{T}_{\Phi_{ro}}$, then semiregularization $\mathcal{T}_{\Phi_{ro}}$ is the coarsest topology such that

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Corollary 3.

Let $\Phi \in \mathbf{\Phi}(\mathbb{R}, \tau_0)$ and for every set $A \in \tau_o$ every $x \in \Phi(A)$ be the both-side accumulation point of $\Phi(A)$. If for every τ_0-regular open set V and for every $x \in \Phi(V)$ there exists at x an interval closed set $F \subset V$ such that $x \in \Phi(\mathrm{int}(F))$, then semiregularization $\mathcal{T}_{\Phi_{ro}}$ is the coarsest topology such that

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The above theorem has application for operators:

- $\Phi_d \in \mathbf{\Phi}(\mathbb{R}, \tau_o)$ (W. Wojdowski);
- $\Phi_I \in \mathbf{\Phi}(\mathbb{R}, \tau_o)$ (E. Łazarow, W. Poreda, E. Wagner-Bojakowska);
- $\Phi_\psi \in \mathbf{\Phi}(\mathbb{R}, \tau_o)$ (W. Wilczyński, W. Wojdowski);
- $\Phi_{I(J)} \in \mathbf{\Phi}(\mathbb{R}, \tau_o)$ (R. Wiertelak).

Let $A \in \mathcal{L}$ and $J = \{J_n\}_{n\in\mathbb{N}}$ be a sequence of intervals tending to zero. It means that

$$|J_n| \underset{n\to\infty}{\longrightarrow} 0 \wedge c(J_n) \underset{n\to\infty}{\longrightarrow} 0,$$

where $c(J_n)$ is a center of interval J_n.

$$x_0 \in \Phi_J(A) \text{ iff } \lim_{n\to\infty} \frac{\lambda(A \cap (J_n + x_0))}{|J_n|} = 1.$$

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Theorem 9 (R. Wiertelak).

For every τ_0-regular open set V and for every $x \in \Phi_J(V)$ there exists at x an interval closed set $F \subset V$ such that $x \in \Phi_J(\mathrm{int}(F))$.

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For every τ_0-regular open set V and for every $x \in \Phi_J(V)$ there exists at x an interval closed set $F \subset V$ such that $x \in \Phi_J(\mathrm{int}(F))$.

Theorem 10.

If $\Phi = \Phi_J$, then $\Phi \in \mathbf{\Phi}(\mathbb{R}, \tau_o)$,$\tau_0 \subset \mathcal{T}_{\Phi_{ro}}$ and semiregularization $\mathcal{T}_{\Phi_{ro}}$ is the coarsest topology such that

$$\mathcal{C}(\langle\mathbb{R}, \mathcal{T}_{\Phi_{ro}}\rangle) = \mathcal{C}(\langle\mathbb{R}, \mathcal{T}_{\Phi_r}\rangle).$$

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