DIFFERENCE OF TWO STRONG ŚWIATKOWSKI LOWER SEMICONTINUOUS FUNCTIONS

ROBERT MENKYNA

The aim of this lecture is the solution of the problem: characterize the difference of lower semicontinuous strong Swiatkowski functions (raised by A. Maliszewski).

We deal with the classes of real functions defined on the real line and we will use the following symbols for the classes of functions:

symbol	classes of functions
C	continuous
D	Darboux
Q	quasi − continuous
lsc	lower semicontinuous
usc	upper semicontinuous
$\acute{S}_s$	strong Światkowski

$\acute{S}_s lsc$ denotes $\acute{S}_s \cap lsc$ and analogical meaning have designations $Dlsc$ and $DQlsc$. We will denote $\hat{\mathfrak{B}}_1$ or $lsc - lsc$ respectively, the family of all differences of two lower (uper) semicontinuous functions. Again, analogous meaning have designation $DQlsc - DQlsc$ or $\acute{S}_s lsc - \acute{S}_s lsc$ respectively.

The real line is denoted by $\mathbb{R}$ and let $I \subset \mathbb{R}$ be an interval.

Definition 1. A function $f : I \to \mathbb{R}$ is the *Darboux function* if it maps connected sets onto conected sets.

A function f is said to be *quasi − continuous* at the point x_0 if for each open set $U \ni x_0$ and each open set $V \ni f(x_0)$ there is a nonempty open set $W \subset U$ with $f(W) \subset V$. f is *quasi − continuous functions* if it is quasi-continuous at each point $x \in I$.

We say that f is *strong Światkowski function* if whenever $\alpha, \beta \in I$, $\alpha < \beta$, and $y \in (f(\alpha), f(\beta))$, there is an $x_0 \in (\alpha, \beta) \cap C_f$ such that $f(x_0) = y$.

It is evident that

$$\acute{S}_s lsc \subset DQlsc.$$

A. Maliszewski in $[1], page107$, formulate following open problem: Characterize the sums and the differences of upper senicontinuous strong Światkowski functions.

A. Maliszewski in [3] and [2] solve the analogous problem for the family of Darboux upper semicontinuous quasi-continuous functions. It is proved in [3] that each upper semicontinuous function is the sum of two Darboux upper semicontinuous quasi-continuous functions. Moreover, later it is proved in [4] that each upper

Date: 23.4.2014.

semicontinuous function is the sum of two upper semicontinuous strong Światkowski functions, that is

$$usc = DQusc + DQusc = \acute{S}_s usc + \acute{S}_s usc$$

With respect to the difference of two upper senicontinuous functions it is proved in [2] that a Darboux quasi-continuous function $f \in \hat{\mathfrak{B}}_1$ if and only if it is equal to the difference of two Darbox quasi-continuous upper semicontinuous functions. This assertion may be represented by the equation

$$DQ\,(usc - usc) = DQusc - DQusc \qquad (\star)$$

The next Theorem 2 is know (see [1], Theorem 3.3., p. 96).

Theorem 2. *Let the function $f \in DQlsc$ and let $\varepsilon > 0$ be arbitrary real number. Then there exists a continuous function g, $0 \le g \le \varepsilon$ such, that the function $f + g \in \acute{S}_s lsc$.*

One can, however, prove more:

Theorem 3. *Let the functions $f_1, f_2, ..., f_m \in DQlsc$ and let $\varepsilon > 0$ be arbitrary real number. Then there exists a continuous function g, $0 \le g \le \varepsilon$, such that the function $f_i + g \in \acute{S}_s lsc$ for every $i = 1, 2, ..., m$.*

Since $\acute{S}_s lsc \subset lscDQ$, so

$$\acute{S}_s lsc - \acute{S}_s lsc \subset lscDQ - lscDQ.$$

If a function $f = f_1 - f_2$, where $f_1, f_2 \in DQlsc$, then according to Theorem 3 there exists a continuous function α such that the functions $f_1+\alpha, f_2+\alpha$ are from the class $\acute{S}_s lsc$ and therefore the function $f = f_1 - f_2 = (f_1 + \alpha) - (f_2 + \alpha) \in \acute{S}_s lsc - \acute{S}_s lsc$. We prove that

$$\acute{S}_s lsc - \acute{S}_s lsc \supset DQlsc - DQlsc.$$

From foregoing it follows, that

$$(lsc - lsc) \cap DQ = DQlsc - DQlsc = \acute{S}_s lsc - \acute{S}_s lsc,$$

which characterize the class of differences of lower (upper) semicontinuous strong Światkowski functions.

Remark 4. Suppose now that the function

$$f \in \left\{ \sum_{n=1}^{\infty} f_n : \sum_{n=1}^{\infty} |f_n(x)| < \infty \text{ for every } x \in I, \text{ with each } f_n \in C \right\}$$

According [4], this class of functions coincide with the class of differences of lower semicontinuous functions, that is

$$f = f_1 - f_2, \text{ where } f_1, f_2 \in lsc.$$

We are interested which properties of the function f result from the properties of functions f_1 and f_2 and conversely.

$$\begin{array}{lcl} f & & f_1, f_2 \\ \textit{boundedness} & \Leftarrow & \textit{boundedness} \\ \textit{continuity at } x_0 \in I & \Leftarrow & \textit{continuity at } x_0 \in I \\ \textit{boundedness} & \nRightarrow & \textit{boundedness} \\ \textit{continuity a.e.} & \nRightarrow & \textit{continuity a.e.} \\ DQ & \Leftrightarrow & DQ \\ \acute{S}_s & \nLeftarrow & \acute{S}_s \end{array}$$

References

[1] A. Maliszewski,*Darboux Property and Quasi-continuity: A Uniform Approach*, Wydaw. Uczelniane WSP, 1996

[2] A. Maliszewski, *On the differences of upper semicontinuous quasi-continuous functions*, Math. Slovaca, Vol. 48 (1998),No. 3, 245-252.

[3] A. Maliszewski, *On the sums of Darboux upper semicontinuous quasi-continuous functions,* Real Anal. Exch., 20 (1994-95), 244-249.

[4] R. Menkyna, *On the sums of lower semicontinuous strong Światkowski functions*, Real. Anal. Exch., 39 (1), 2013/2014, 15-32.

[4] W. Sierpiński, *Sur les fonctions développables en séries absolument convergentes de fonctions continues*, Fund. Math. 2 (1921), p. 15-27

Institute of Aurel Stodola, Faculty of Electrical Engineering, University of Žilina, Liptovský Mikuláš, Slovakia

E-mail address: menkyna@lm.uniza.sk