# On openness of bilinear mappings

Adam Majchrzycki

Łódz University of Technology, Poland

28th Summer Conference on Real Functions Theory, Stara Lesna, 2014

## Definitions

Let $X$ and $Y$ be metric spaces. A mapping $T\colon X \to Y$ is called

- open if it maps every open set onto an open set;
- open at a point $x \in X$ if $T(x) \in \operatorname{int} T[V]$ for each neighbourhood $V$ of $x$.

## Fact

A mapping $T\colon X \to Y$ is open if and only if it is open at every point.

## Theorem (Banach openness principle)

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The bilinear analogue of the Banach openness principle is false.

Indeed,

• W. Rudin: The mapping $T\colon \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$, T(t,x,y):=(tx,ty), is not open at $(0,1,1)$;

• P.J. Cohen (1974) and C. Horowitz (1975) found bilinear continuous surjections that are not open at 0 (finite dimensional, and infinite dimensional cases).

• Multiplication in $C[0,1]$ is not open, for instance at $(f,f)$ where $f(x)=x-1/2$; D.H. Fremlin (oral communication) (see also, M. Balcerzak, A. Wachowicz, W. Wilczyński, 2005).

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We have obtained some other positive results.

**Theorem (M.Balcerzak, A. M., A. Wachowicz, 2013)**

Let $(X, \mathcal{S}, \mu)$ be a measure space with $\mu(X) > 0$. Then the multiplication from $L_p \times L_q$ to $L_1$ (for $p \in [1, \infty]$, $1/p + 1/q = 1$) is an open continuous bilinear surjection. In particular, this holds for multiplication from $\ell_p \times \ell_q$ to $\ell_1$. Also, the multiplication from $\ell_1 \times c_0$ to $\ell_1$ is an open bilinear surjection.

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Let $X$ and $Y$ be metric spaces and let $f\colon X \to Y$.

**Remark**

The mapping $f$ is open if and only if

$$\forall x \in X \, \forall \varepsilon > 0 \, \exists \delta > 0 \; B(f(x), \delta) \subset f[B(x, \varepsilon)].$$

We introduce a stronger notion of openness.

**Definition**

The mapping $f$ is called *uniformly open* whenever

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Note that

- The function $f\colon \mathbb{R} \to (-\pi/2, \pi/2)$, $f(x) = \arctan x$, is open but not uniformly open.
- Every continuous linear operator from a Banach space onto a Banach space is a uniformly open mapping.

**Fact (folklore?)**

Let $X, Y$ be Banach spaces. Then every nonzero bilinear continuous functional $T\colon X \times Y \to \mathbb{R}$ is a surjective open mapping.

**Proposition (M.Balcerzak, A. M., F. Strobin)**

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## Definition

We will say that $(X, \mathcal{S}, \mu)$ is a topological measure space if $X$ is a topological space with all Borel sets belonging to $\mathcal{S}$.

We will assume that $X$ is $\sigma$-compact.

Let $L_\infty^0$ denote the space of all functions $f \in L_\infty$ which vanish at infinity, i.e.
for any $\varepsilon > 0$, there is a compact set $K \subset X$ such that
$\mu(\{x \in X \setminus K : |f(x)| > \varepsilon\}) = 0$.

Then $L_\infty^0$ is a closed linear subspace of $L_\infty$ and so, $L_\infty^0$ with the norm $||\cdot||_\infty$ is a Banach space.

## Theorem (M.Balcerzak, A. M., F. Strobin)

Assume that $(X, \mathcal{S}, \mu)$ is a topological measure space where $X$ is $\sigma$-compact. Then the multiplication from $L_1 \times L_\infty^0$ to $L_1$ is a continuous surjection which is a uniformly open mapping.

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For $X = \mathbb{N}$, $\mathcal{S} = \mathcal{P}(\mathbb{N})$ and the counting measure, we get

**Corollary**

The multiplication from $\ell_1 \times c_0$ to $\ell_1$ is a uniformly open mapping.

Note that the statement of the Theorem remains true if we assume there is a $\sigma$-compact subset $Y$ of $X$ such that $\mu(X \setminus Y) = 0$. We then call $X$ *almost $\sigma$-compact*.

However, if we drop the assumption of the almost $\sigma$-compactness of $X$, the surjectivity of multiplication can be lost.

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## Example

Let $X = [0,1)$ with the Sorgenfrey topology and the Lebesgue measure. Then the Borel sets are the same as in the natural topology, and all compact sets are countable. Hence this measure space is not almost $\sigma$-compact.

Let $h(x) := 1$ for every $x \in X$. Then $h \in L_1$, and if $h = fg$ then $g(x) \neq 0$ for every $x \in X$.

Suppose that $g \in L_\infty^0$. Then for each $n \in \mathbb{N}$ there is a compact set $K_n$ such that $\mu(\{x \in X \setminus K_n \colon |g(x)| > 1/n\}) = 0$. Hence

$$X = \{x \in X \colon g(x) \neq 0\} = \bigcup_{n\in\mathbb{N}} K_n \cup \bigcup_{n\in\mathbb{N}} \{x \in X \setminus K_n \colon |g(x)| > 1/n\}.$$

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Let $X = [0,1)$ with the Sorgenfrey topology and the Lebesgue measure. Then the Borel sets are the same as in the natural topology, and all compact sets are countable. Hence this measure space is not almost $\sigma$-compact.

Let $h(x) := 1$ for every $x \in X$. Then $h \in L_1$, and if $h = fg$ then $g(x) \neq 0$ for every $x \in X$.

Suppose that $g \in L^0_\infty$. Then for each $n \in \mathbb{N}$ there is a compact set $K_n$ such that $\mu(\{x \in X \setminus K_n \colon |g(x)| > 1/n\}) = 0$. Hence

$$X = \{x \in X \colon g(x) \neq 0\} = \bigcup_{n\in\mathbb{N}} K_n \cup \bigcup_{n\in\mathbb{N}} \{x \in X \setminus K_n \colon |g(x)| > 1/n\}.$$

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Thank you for your attention.