Subsets of some families of real functions and their algebrability

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Bibliography

J. Wódka, *Subsets of some families of real functions and their algebrability*, Linear Algebra Appl., 459 (2014), 454-464, JCR

Plan

1. Definitions
2. Examples
3. Main results
4. Open problems

Definitions

Let $f\colon \mathbb{R} \to \mathbb{R}$.

- Function f is *quasi-continuous* if for all $a < x < b$ and each $\varepsilon > 0$ there exists a nondegenerate interval $J \subset (a, b)$ such that $\operatorname{diam} f[J \cup \{x\}] < \varepsilon$. The family of all quasi-continuous functions will be denoted by $\mathcal{Q}$.
- Function f is *Świątkowski* if for all $a < b$ with $f(a) \neq f(b)$, there is a y between $f(a)$ and $f(b)$ and an $x \in (a, b) \cap \mathcal{C}(f)$ such that $f(x) = y$. The family of all Świątkowski functions will be denoted by $\acute{\mathcal{S}}$.
- Function f is *strong Świątkowski* if for all $a < b$ and each y between $f(a)$ and $f(b)$, there is an $x \in (a, b) \cap \mathcal{C}(f)$ with $f(x) = y$. The family of all strong Świątkowski functions will be denoted by $\acute{\mathcal{S}}_s$.
- The family of all Darboux functions will be denoted by $\mathcal{D}$.

What we should know

Inclusions

The followin inclusions hold:

$$\acute{\mathcal{S}}_s \subsetneq \acute{\mathcal{S}},$$
$$\acute{\mathcal{S}}_s \subsetneq \mathcal{Q},$$
$$\acute{\mathcal{S}}_s \subsetneq \mathcal{D},$$

The families $\acute{\mathcal{S}}$, $\mathcal{Q}$ and $\mathcal{D}$ are incomparable.

Theorem (R.Pawlak, 1985)

Let $I \subset \mathbb{R}$ be an interval. Then the following equalities

$$\mathcal{D} \cap \mathcal{Q} \cap \acute{\mathcal{S}} = \mathcal{D} \cap \mathcal{Q} = \mathcal{D} \cap \acute{\mathcal{S}}$$

hold.

Examples

- $\mathcal{D} \cap \mathcal{Q} \cap \acute{\mathcal{S}} \setminus \acute{\mathcal{S}}_s$

$$f(x) = \begin{cases} \operatorname{sgn} x \cdot e^{-|x|} \cdot \max\{|x|, \sin 1/x\}, & \text{when } x > 0 \\ 0, & \text{when } x = 0. \end{cases}$$

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- $\mathcal{D} \setminus (\mathcal{Q} \cup \acute{\mathcal{S}})$

Nowhere continuous Darboux functions.

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- $\mathcal{Q} \setminus (\mathcal{D} \cup \acute{\mathcal{S}})$

$$f(x) = \begin{cases} 0, & \text{when } x \geq 0 \\ 1, & \text{when } x < 0. \end{cases}$$

Examples

- $\mathcal{Q} \cap \acute{\mathcal{S}} \setminus \mathcal{D}$

$$f(x) = \begin{cases} x+1, & \text{when } x \geq 0 \\ x-1, & \text{when } x < 0. \end{cases}$$

Examples

- $\mathcal{Q} \cap \acute{\mathcal{S}} \setminus \mathcal{D}$

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- $\acute{\mathcal{S}} \setminus (\mathcal{D} \cup \mathcal{Q})$

$$f(x) = \begin{cases} x+1, & \text{when } x > 0, \\ 0, & \text{when} x = 0, \\ x-1, & \text{when } x < 0. \end{cases}$$

Definitions

- Let $\mathcal{L}$ be a vector space and a set $A \subset \mathcal{L}$. We say that A is κ-lineable if $A \cup \{\theta\}$ contains a κ-dimensional vector space.
- Let $\mathcal{L}$ be a linear commutative algebra and a set $A \subset \mathcal{L}$. We say that A is κ-algebrable if $A \cup \{\theta\} \subset \mathcal{L}$ contains a κ-generated algebra B (i.e. the minimal cardinality of the set of generators of B is equal to κ).
- Let $\mathcal{L}$ be a commutative algebra and a set $A \subset \mathcal{L}$. We say that A is strongly κ-algebrable if $A \cup \{\theta\}$ contains a κ-generated algebra that is isomorphic to a free algebra.

Main results

$\mathcal{D}$	$\mathcal{Q}$	$\acute{\mathcal{S}}$	$\acute{\mathcal{S}}_s$	level of algebrability
+	+	+	-	strongly $\mathfrak{c}$-algebrable
+	+	-	-	————
+	-	+	-	————
+	-	-	-	strongly $2^{\mathfrak{c}}$-algebrable
-	+	+	-	strongly $\mathfrak{c}$-algebrable
-	+	-	-	no algebrable
-	-	+	-	strongly $\mathfrak{c}$-algebrable
-	-	-	-	$2^{\mathfrak{c}}$-algebrable

What we should know

Theorem

The set of quasi-continuous functions is not $\mathfrak{c}^+$-lineable.

Theorem

The set of Świątkowski functions is not $\mathfrak{c}^+$-lineable.

Theorem (Bartoszewicz, Głąb, Paszkiewicz, 2013)

The family of nowhere continuous Darboux functions is strongly $2^{\mathfrak{c}}$-algebrable.

Theorem (Bartoszewicz, Bienias, Głąb, 2012)

The family of everywhere discontinuous functions, that have finitely many values, is $2^{\mathfrak{c}}$-algebrable but not strongly 1-algebrable.

Main results

$\mathcal{D}$	$\mathcal{Q}$	$\acute{\mathcal{S}}$	$\acute{\mathcal{S}}_s$	level of algebrability
+	+	+	-	strongly $\mathfrak{c}$-algebrable
+	+	-	-	____________
+	-	+	-	____________
+	-	-	-	strongly $2^{\mathfrak{c}}$-algebrable
-	+	+	-	strongly $\mathfrak{c}$-algebrable
-	+	-	-	no algebrable
-	-	+	-	strongly $\mathfrak{c}$-algebrable
-	-	-	-	$2^{\mathfrak{c}}$-algebrable

Useful theorem

Definition

We say that a function $f : \mathbb{R} \to \mathbb{R}$ is exponential-like (of the rank m) if

$$f(x) = \sum_{i=1}^{m} a_i e^{\beta_i x},$$

for $\beta_i \in \mathbb{R} \setminus \{0\}$, $\beta_i \neq \beta_j$ for $i \neq j$ and $a_i \in \mathbb{R} \setminus \{0\}$, $i \in \{1, 2, \ldots, m\}$.

The family of all exponential-like functions of rank m will be denoted by $\mathcal{E}_m$ and $\mathcal{E} := \bigcup_{m \in \mathbb{N}} \mathcal{E}_m$.

Theorem (Balcerzak, Bartoszewicz, Filipczak, 2013)

Given a family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$, assume that there exists a function $F \in \mathcal{F}$ such that $f \circ F \in \mathcal{F} \setminus \{\theta\}$ for every $f \in \mathcal{E}$. Then $\mathcal{F}$ is strongly $\mathfrak{c}$-algebrable.

Skech of the proof

Lemma (Balcerzak, Bartoszewicz, Filipczak, 2013)

For every positive integer m, any $f \in \mathcal{E}_m$ and each $c \in \mathbb{R}$, the preimage $f^{-1}[\{c\}]$ has at most m elements. In particular there exists a finite decomposition of $\mathbb{R}$ into intervals, such that f is strictly monotone on each of them.

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Świątkowski function

Function f is *Świątkowski* if for all $a < b$ with $f(a) \neq f(b)$, there is a y between $f(a)$ and $f(b)$ and an $x \in (a, b) \cap \mathcal{C}(f)$ such that $f(x) = y$.

Other cases

Cantorval-informal definition

The cantorval we call a set which is a union of the Cantor set and all intervals removed in every second step in the construction of the Cantor set.

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Theorem

The family $\acute{\mathcal{S}} \setminus (\mathcal{D} \cup \mathcal{Q})$ *is not* 1*-algebrable.*

Proof.

- Show that it has to be some *gap*,
- Show that for each $f \in \acute{\mathcal{S}} \setminus (\mathcal{D} \cup \mathcal{Q})$ there exists polynomial W such that $W \circ f \notin \acute{\mathcal{S}}$.

$\square$

Open problems

Question 1

Is the family $\mathbb{R}^{\mathbb{R}} \setminus (\mathcal{Q} \cup \acute{\mathcal{S}} \cup \mathcal{D})$ strongly $2^{\mathfrak{c}}$-algebrable?

Question 2

Is the family $\acute{\mathcal{S}} \setminus (\mathcal{D} \cup \mathcal{Q})$ lineable?

Thank you for your attention