Resolvability of abstract density topologies in $\mathbb{R}^n$ generated by lower or almost lower density operators

Grażyna Horbaczewska
University of Łódź

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Resolvability

Definition (1943, E. Hewitt)

A topological space (X, τ) is resolvable if it contains two disjoint dense subsets.

Definition

Let α be an arbitrary cardinal number greater than one.
We say that a space (X, τ) is α-resolvable if there is a family of α-many pairwise disjoint dense sets each of which intersects each nonempty open subsets of X in at least α points.

$$\triangle(X, \tau) := min\{card(A) : A \in \tau \wedge A \neq \emptyset\}$$

Definition

(X, τ) is maximally resolvable if it is $\triangle(X, \tau)$-resolvable.

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Maximal resolvability

Theorem (W.W.Comfort, S. Garcia-Ferreira)

If a Hausdorff space without isolated points is locally compact or metrizable then it is maximally resolvable.

Extraresolvability

Definition

(X, τ) is extraresolvable if there exists a family $\mathcal{M}$ of dense subsets of X such that

$$card(\mathcal{M}) > \triangle(X, \tau)$$

and for every $C, D \in \mathcal{M}$, $C \neq D$, the set $C \cap D$ is nowhere dense.

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Let X be a nonempty set, $\mathcal{S}$ be a σ-algebra of sets from X and $\mathcal{I} \subset \mathcal{S}$ be a proper σ-ideal.

Definition

We say that an operator $\Phi : \mathcal{S} \to \mathcal{S}$ is a lower density operator on a measurable space $(X, \mathcal{S}, \mathcal{I})$ if
(i) $\Phi(\emptyset) = \emptyset$, $\Phi(X) = X$;
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Abstract density topologies

Definition

We say that a topology τ is an abstract density topology on X if there exists a lower density operator Φ on $(X, \mathcal{S}, \mathcal{I})$ such that

$$\tau = \mathcal{T}_\Phi,$$

where $\mathcal{T}_\Phi := \{A \in \mathcal{S} : A \subset \Phi(A)\}$.
Topology $\mathcal{T}_\Phi$ is called generated by a lower density operator on $(X, \mathcal{S}, \mathcal{I})$.

Theorem

Let Φ be a lower density operator on $(X, \mathcal{S}, \mathcal{I})$.
Then the family $\mathcal{T}_\Phi$ is a topology on X if and only if the pair $(\mathcal{S}, \mathcal{I})$ has the hull property.

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We say that an operator $\Phi : \mathcal{S} \to 2^X$ is an almost lower density operator on a measurable space $(X, \mathcal{S}, \mathcal{I})$ if
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(iv-a) $\forall_{A\in\mathcal{S}}\ \Phi(A)\setminus A \in \mathcal{I}$.

Theorem

Let Φ be an almost lower density operator on $(X, \mathcal{S}, \mathcal{I})$. If the pair $(\mathcal{S}, \mathcal{I})$ has the hull property, then the family $\mathcal{T}_\Phi = \{A \in \mathcal{S} : A \subset \Phi(A)\}$ is a topology on X.

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$$(A \in \mathcal{T}_\Phi \ \wedge \ A \neq \emptyset) \ \Rightarrow \ A \notin \mathcal{I}$$

Theorem

If every set $A \in \mathcal{S} \setminus \mathcal{I}$ contains a perfect set ...

$\mathcal{I}$ - Borel and $\mathcal{S} = \mathcal{B}or \triangle \mathcal{I}$ $(\mathcal{I} = \mathcal{N} \cap \mathcal{K})$
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$(\mathbb{R}^n, \mathcal{S}, \mathcal{I})$ $(\mathbb{R}^n, \mathcal{T}_\Phi)$-generated by ALD

$$(A \in \mathcal{T}_\Phi \ \wedge \ A \neq \emptyset) \ \Rightarrow \ A \notin \mathcal{I}$$

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If every set $A \in \mathcal{S} \setminus \mathcal{I}$ contains a perfect set then

- $\triangle(\mathbb{R}^n, \mathcal{T}_\Phi) = \mathfrak{c}$,
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There exists a partition of $\mathbb{R}^n$ into continuum many Bernstein sets.

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Any topological space $(\mathbb{R}^n, \mathcal{T}_\Phi)$, where $\mathcal{T}_\Phi$ is a topology generated by ALD on $(\mathbb{R}^n, \mathcal{S}, \mathcal{I})$ with a property that every $A \in \mathcal{S} \setminus \mathcal{I}$ contains a perfect set, is maximally resolvable.

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Under MA in $\mathbb{R}^n$ there exists a family of cardinality $2^{\mathfrak{c}}$ almost disjoint Bernstein sets.

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ZFC+CPA (Covering Property Axiom) and an additional assumption: $2^{\omega_1} = \omega_2$.

Consequences of CPA:

- $\mathfrak{c} = \omega_2$
- $cf(\mathcal{N}) = \omega_1$

$$cf(\mathcal{I}) := min\{card(\mathcal{I}_0) : \mathcal{I}_0 \subset \mathcal{I} \wedge \forall_{J \in \mathcal{I}} \exists_{J_0 \in \mathcal{I}_0} J \subset J_0\}$$

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?

Does a model of ZFC exist in which the space $(\mathbb{R}^n, \mathcal{T}_\Phi)$, where $\mathcal{T}_\Phi$ is a topology generated by an almost lower density operator on $(\mathbb{R}^n, \mathcal{L}, \mathcal{N})$ or on $(\mathbb{R}^n, \mathcal{B}, \mathcal{K})$, is not extraresolvable?

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