On O'Malley ϱ-upper continuous functions

Stanisław Kowalczyk & Katarzyna Nowakowska

Institute of Mathematics
Pomeranian Academy in Słupsk

Stará Lesná 2014

ϱ-upper continuous functions

- Definition ([4])

  Let E be a measurable subset of $\mathbb{R}$, let $x \in \mathbb{R}$ and let $0 < \varrho < 1$. We say that the point x is a point of ϱ-type upper density of E when $\overline{d}(E,x) > \varrho$.

  - According to [3], $\overline{d}(E,x) = \max\{\overline{d}^{-}(E,x), \overline{d}^{+}(E,x)\}$, where $\overline{d}^{+}(E,x) = \limsup_{t\to 0^+} \frac{\lambda(E\cap[x,x+t])}{t}$ and $\overline{d}^{-}(E,x) = \limsup_{t\to 0^+} \frac{\lambda(E\cap[x-t,x])}{t}$).

ϱ-upper continuous functions

- **Definition ([4])**

  Let E be a measurable subset of $\mathbb{R}$, let $x \in \mathbb{R}$ and let $0 < \varrho < 1$. We say that the point x is a point of ϱ-type upper density of E when $\overline{d}(E, x) > \varrho$.

  - According to [3], $\overline{d}(E, x) = \max\{\overline{d}^-(E, x), \overline{d}^+(E, x)\}$, where $\overline{d}^+(E, x) = \limsup_{t \to 0^+} \frac{\lambda(E \cap [x, x+t])}{t}$ and $\overline{d}^-(E, x) = \limsup_{t \to 0^+} \frac{\lambda(E \cap [x-t, x])}{t}$).

- **Definition ([4])**

  Let $\varrho \in (0, 1)$. A function $f : I \to \mathbb{R}$ is said to be ϱ-upper continuous at $x \in I$ if there is a measurable set $E \subset I$ such that the point x is a point of ϱ-type upper density of E, $x \in E$ and $f|E$ is continuous at x. If f is ϱ-upper continuous at every point of I we say simply that f is ϱ-upper continuous.

ϱ-upper continuous functions

- Definition ([4])

  Let E be a measurable subset of $\mathbb{R}$, let $x \in \mathbb{R}$ and let $0 < \varrho < 1$. We say that the point x is a point of ϱ-type upper density of E when $\overline{d}(E, x) > \varrho$.

  - According to [3], $\overline{d}(E, x) = \max\{\overline{d}^{-}(E, x), \overline{d}^{+}(E, x)\}$, where $\overline{d}^{+}(E, x) = \limsup_{t \to 0^+} \frac{\lambda(E \cap [x, x+t])}{t}$ and $\overline{d}^{-}(E, x) = \limsup_{t \to 0^+} \frac{\lambda(E \cap [x-t, x])}{t}$).

- Definition ([4])

  Let $\varrho \in (0, 1)$. A function $f : I \to \mathbb{R}$ is said to be ϱ-upper continuous at $x \in I$ if there is a measurable set $E \subset I$ such that the point x is a point of ϱ-type upper density of E, $x \in E$ and $f|E$ is continuous at x. If f is ϱ-upper continuous at every point of I we say simply that f is ϱ-upper continuous.

  - We will denote the class of all ϱ-upper continuous functions by $\mathcal{U}C_{\varrho}$.

- **Theorem ([5])**

  Let $0 < \varrho < 1$ and let $f: I \to \mathbb{R}$ be a measurable function. Then f is ϱ-upper continuous at $x \in I$ if and only if

  $$\lim_{\varepsilon \to 0^+} \overline{d}\big(\{y \in I : |f(x) - f(y)| < \varepsilon\}, x\big) > \varrho.$$

O'Malley ϱ-upper continuous functions

- Definition

  Let $\varrho \in (0,1)$. A point x_0 is said to be a point of ϱ-type upper density in O'Malley sense on the right of a measurable set E if for each $\varepsilon > 0$ there exists $\delta \in (0, \varepsilon)$ such that the inequality

  $$\frac{\lambda\left(E \cap [x_0, x_0 + \delta]\right)}{\delta} > \varrho$$

  holds.

  - Similarly, we define left-sided O'Malley ϱ-upper density and x_0 is a point of ϱ-type upper density in O'Malley sense of a measurable set E if and only if it is a point of ϱ-type upper density in O'Malley sense of E on the right or on the left.

- Definition

  Let $\varrho \in (0,1)$. A function $f\colon I \to \mathbb{R}$ is said to be ϱ-upper continuous in O'Malley sense at $x_0 \in I$ (f is O'Malley ϱ-upper continuous at x_0, in abbreviation) if there exists a measurable set $E \subset I$ containing x_0 such that x_0 is a point of ϱ-type upper density in O'Malley sense of E and $f|E$ is continuous at x_0. A function f is said to be O'Malley ϱ-upper continuous if it is O'Malley ϱ-upper continuous at each point $x \in I$.

- **Definition**
  Let $\varrho \in (0,1)$. A function $f\colon I \to \mathbb{R}$ is said to be ϱ-upper continuous in O'Malley sense at $x_0 \in I$ (f is O'Malley ϱ-upper continuous at x_0, in abbreviation) if there exists a measurable set $E \subset I$ containing x_0 such that x_0 is a point of ϱ-type upper density in O'Malley sense of E and $f|E$ is continuous at x_0. A function f is said to be O'Malley ϱ-upper continuous if it is O'Malley ϱ-upper continuous at each point $x \in I$.
  - We denote the class of all O'Malley ϱ-upper continuous functions by $\mathcal{OUC}_\varrho$.

- Corollary

  Let $0 < \varrho_1 < \varrho_2 < 1$, $f\colon I \to \mathbb{R}$ and $x_0 \in I$. If f is ϱ_2-upper continuous in O'Malley sense at x_0, then f is ϱ_1-upper continuous at x_0 in O'Malley sense.

- Corollary

  Let $0 < \varrho_1 < \varrho_2 < 1$, $f : I \to \mathbb{R}$ and $x_0 \in I$. If f is ϱ_2-upper continuous in O'Malley sense at x_0, then f is ϱ_1-upper continuous at x_0 in O'Malley sense.

- Corollary

  Let $0 < \varrho < 1$, $f : I \to \mathbb{R}$ and $x_0 \in I$. If f is ϱ-upper continuous at x_0, then f is ϱ-upper continuous in O'Malley sense at x_0.

- Theorem

Let $\varrho \in (0,1)$. A measurable function $f\colon I \to \mathbb{R}$ is ϱ-upper continuous in O'Malley sense at $x_0 \in I$ if and only if for each $\varepsilon > 0$ and $\delta > 0$ there exists $\eta < \delta$ such that

$$\frac{\lambda(\{x\colon |f(x) - f(x_0)| < \varepsilon\} \cap [x_0, x_0 + \eta])}{\eta} > \varrho$$

or

$$\frac{\lambda(\{x\colon |f(x) - f(x_0)| < \varepsilon\} \cap [x_0 - \eta, x_0])}{\eta} > \varrho.$$

O'Malley [0]-upper continuous function

- Definition
  A function $f : I \to \mathbb{R}$ *is called* $[0]$*-upper continuous in O'Malley sense at* $x_0 \in I$ *(*f *is O'Malley* $[0]$*-upper continuous at* x_0*, in abbreviation) if there exists* $\varrho > 0$ *such that* f *is O'Malley* ϱ*-upper continuous at* x_0*.*

O'Malley [0]-upper continuous function

- Definition
  A function $f : I \to \mathbb{R}$ *is called* [0]*-upper continuous in O'Malley sense at* $x_0 \in I$ *(*f *is O'Malley* [0]*-upper continuous at* x_0*, in abbreviation) if there exists* $\varrho > 0$ *such that* f *is O'Malley* ϱ*-upper continuous at* x_0*.*
  - We denote the class of all O'Malley [0]-upper continuous functions by $\mathcal{OUC}_{[0]}$.

O'Malley [0]-upper continuous function

- Definition
  A function $f : I \to \mathbb{R}$ is called [0]*-upper continuous in O'Malley sense at $x_0 \in I$ (f is O'Malley* [0]*-upper continuous at x_0, in abbreviation) if there exists $\varrho > 0$ such that f is O'Malley ϱ-upper continuous at x_0.*
  - We denote the class of all O'Malley [0]-upper continuous functions by $\mathcal{OUC}_{[0]}$.
- Corollary
  $\bigcup_{\varrho \in (0,1)} \mathcal{OUC}_\varrho \subsetneq \mathcal{OUC}_{[0]}$.

O'Malley [0]-upper continuous function

- Definition
  A function $f : I \to \mathbb{R}$ is called $[0]$*-upper continuous in O'Malley sense at* $x_0 \in I$ *(f is O'Malley* $[0]$*-upper continuous at x_0, in abbreviation) if there exists $\varrho > 0$ such that f is O'Malley ϱ-upper continuous at x_0.*
  - We denote the class of all O'Malley [0]-upper continuous functions by $\mathcal{OUC}_{[0]}$.
- Corollary
  $\bigcup_{\varrho\in(0,1)} \mathcal{OUC}_\varrho \subsetneq \mathcal{OUC}_{[0]}$.
- Theorem
  If $f \in \mathcal{OUC}_{[0]}$, then f is measurable.

- Theorem

Let $f : I \to \mathbb{R}$ be a measurable function and let $x_0 \in I$. The following conditions are equivalent

1. *function f is $[0]$-upper continuous in O'Malley sense at x_0,*
2. *there exists measurable set $E \subset I$ such that $x_0 \in E$, $f|E$ is continuous at x_0 and $\overline{d}(E, x_0) > 0$,*
3. $\lim\limits_{\varepsilon \to 0^+} \overline{d}\big(\{x \in I \colon |f(x) - f(x_0)| < \varepsilon\}, x_0\big) > 0.$
4. *there exists $0 < \varrho < 1$ such that f is ϱ-upper continuous at x_0*

Maximal additive families

- Let $\mathcal{F}$ be any family of real valued functions defined on I. The set $\mathcal{M}_a(\mathcal{F}) = \{g : \forall_{f \in \mathcal{F}}\ f + g \in \mathcal{F}\}$ is called a maximal additive family for $\mathcal{F}$.

Maximal additive families

- Let $\mathcal{F}$ be any family of real valued functions defined on I. The set $\mathcal{M}_a(\mathcal{F}) = \{g : \forall_{f \in \mathcal{F}}\ f + g \in \mathcal{F}\}$ is called a maximal additive family for $\mathcal{F}$.

- Theorem
  Let $\varrho \in (0, 1)$. Then $\mathcal{M}_a(\mathcal{OUC}_\varrho) = \mathcal{C}$.

Maximal additive families

- Let $\mathcal{F}$ be any family of real valued functions defined on I. The set $\mathcal{M}_a(\mathcal{F}) = \{g \colon \forall_{f \in \mathcal{F}}\ f + g \in \mathcal{F}\}$ is called a maximal additive family for $\mathcal{F}$.

- Theorem
  Let $\varrho \in (0,1)$. *Then* $\mathcal{M}_a(\mathcal{OUC}_\varrho) = \mathcal{C}$.

- Theorem
  $\mathcal{M}_a(\mathcal{OUC}_{[0]}) = \mathcal{A}$.

Maximal multiplicative families

- Let $\mathcal{F}$ be any family of real valued functions defined on an open interval I. A set $\mathcal{M}_m(\mathcal{F}) = \{g\colon \forall_{f\in\mathcal{F}}\ f\cdot g \in \mathcal{F}\}$ is called a maximal multiplicative family for $\mathcal{F}$.

Maximal multiplicative families

- Let $\mathcal{F}$ be any family of real valued functions defined on an open interval I. A set $\mathcal{M}_m(\mathcal{F}) = \{g \colon \forall_{f \in \mathcal{F}}\ f \cdot g \in \mathcal{F}\}$ is called a maximal multiplicative family for $\mathcal{F}$.
- Let $f \colon I \to \mathbb{R}$. Then
  - $D_{ap}(f)$ is the set of points at which f is not approximately continuous,
  - $D(f)$ is the set of points at which f is discontinuous,
  - $N_f = \{x \in I \colon f(x) = 0\}$.

- Definition

  Let $\mathcal{Y}(\varrho)$ be a family of all measurable functions $f : I \to \mathbb{R}$ such that at each $x_0 \in D(f)$ the following two conditions hold

  (Y1) $f(x_0) = 0$,

  (Y2) x_0 *is a point of ϱ-type upper density of* $\{x \in I : f(x) = 0\}$.

- Definition
  Let $\mathcal{Y}(\varrho)$ be a family of all measurable functions $f: I \to \mathbb{R}$ such that at each $x_0 \in D(f)$ the following two conditions hold
  - (Y1) $f(x_0) = 0$,
  - (Y2) *x_0 is a point of ϱ-type upper density of $\{x \in I: f(x) = 0\}$.*
- Theorem
  Let $\varrho \in (0,1)$. Then $\mathcal{M}_m(\mathcal{OUC}_\varrho) = \mathcal{Y}(\varrho)$.

- Definition

  Let $\mathcal{Y}(\varrho)$ be a family of all measurable functions $f : I \to \mathbb{R}$ such that at each $x_0 \in D(f)$ the following two conditions hold

  (Y1) $f(x_0) = 0$,

  (Y2) x_0 *is a point of ϱ-type upper density of* $\{x \in I : f(x) = 0\}$.

- Theorem

  Let $\varrho \in (0, 1)$. Then $\mathcal{M}_m(\mathcal{OUC}_\varrho) = \mathcal{Y}(\varrho)$.

- Corollary

  Let $\varrho \in (0, 1)$. Then $\mathcal{C} \subsetneq \mathcal{Y}(\varrho)$.

- Definition

  Let $\mathcal{Z}([0])$ be a family of all measurable functions $f: I \to \mathbb{R}$ such that at each $x_0 \in D_{ap}(f)$ the following two conditions hold

  (Z1) $f(x_0) = 0$,

  (Z2) *for each measurable set E such that $E \supset N_f$ and $\overline{d}(E, x_0) > \varrho$ we have*

  $$\lim_{\varepsilon \to 0^+} \overline{d}(E \cap \{x \in I : |f(x) - f(x_0)| < \varepsilon\}, x_0) > 0.$$

- **Definition**
  Let $\mathcal{Z}([0])$ be a family of all measurable functions $f\colon I \to \mathbb{R}$ such that at each $x_0 \in D_{ap}(f)$ the following two conditions hold
  (Z1) $f(x_0) = 0$,
  (Z2) *for each measurable set E such that $E \supset N_f$ and $\overline{d}(E, x_0) > \varrho$ we have*
  $$\lim_{\varepsilon \to 0^+} \overline{d}(E \cap \{x \in I \colon |f(x) - f(x_0)| < \varepsilon\}, x_0) > 0.$$

- **Theorem**
  $\mathcal{M}_m\left(\mathcal{OUC}_{[0]}\right) = \mathcal{Z}([0])$.

- Definition
  Let $\mathcal{Z}([0])$ *be a family of all measurable functions* $f\colon I \to \mathbb{R}$ *such that at each* $x_0 \in D_{ap}(f)$ *the following two conditions hold*
  (Z1) $f(x_0) = 0$,
  (Z2) *for each measurable set* E *such that* $E \supset N_f$ *and* $\overline{d}(E, x_0) > \varrho$ *we have*
  $$\lim_{\varepsilon \to 0^+} \overline{d}(E \cap \{x \in I\colon |f(x) - f(x_0)| < \varepsilon\}, x_0) > 0.$$

- Theorem
  $\mathcal{M}_m\left(\mathcal{OUC}_{[0]}\right) = \mathcal{Z}([0])$.

- Corollary
  $\mathcal{A} \subsetneq \mathcal{Z}([0])$.

[1] J. Borsík, *Some classes of strongly quasicontinuous functions*, Real Anal. Exchange 30 (2004-2005), 689-702

[2] J. Borsík, *Some properties of porouscontinuous functions*, Math. Slovaca 64 (2014), No. 3, 741-750

[3] A. M. Bruckner, R. J. O'Malley and B. S. Thomson, *Path Derivatives: A Unified View of Certain Generalized Derivatives.*, Trans. Amer. Math. Soc., Vol. 283, (1984), 97-125

[4] S. Kowalczyk, K. Nowakowska *A note on ϱ-upper continuous functions.* Tatra Mt. Math. Publ. 44 (2009), 153-158

[5] S. Kowalczyk, K. Nowakowska *Maximal classes for ϱ-upper continuous functions.* Journal of Applied Analysis, Volume 19, Issue 1 (2013), 69-90

[6] S. Kowalczyk, K. Nowakowska *A note on the* [0]*-lower continuous functions.* Tatra Mt. Math. Publ. 58 (2014), 111-128

[7] R.J. O'Malley, *Note about preponderantly continuous functions.* Revue Roumanie Math. Pureed Appl. 21, (1976), 335-336.