Selections of set-valued functions satisfying a generalized linear inclusion

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Theorem 1 (K. Nikodem, D. Popa (2009), M. Piszczek (2013))

Let K be a convex cone in a real vector space X, $(Y, \|\cdot\|)$ a real Banach space and $\alpha, \beta, p, q > 0$. Consider a set-valued function $F\colon K \to ccl(Y)$ such that

$$\sup\{\operatorname{diam} F(x),\ x \in K\} < +\infty,$$

and

$$\alpha F(x) + \beta F(y) \subset F(px + qy), \quad x, y \in K.$$

1. *If $\alpha + \beta < 1$, then there exists a unique selection $f\colon K \to Y$ of the multifunction F such that*

$$\alpha f(x) + \beta f(y) = f(px + qy),\ x, y \in K.$$

2. *If $\alpha + \beta > 1$, then F is single-valued.*

Let X be a real vector space. A set $K \subset X$ is a *convex cone* iff

1. $\lambda \cdot K \subset K,\ \lambda \geqslant 0,$
2. $K + K \subset K$

Remark 2

Let $\alpha, \beta \neq 0$ and $p, q > 0$. Assume that X and Y are vector spaces and K is a convex cone in X. If $f\colon K \to Y$ satisfies the equation

$$\alpha f(x) + \beta f(y) = f(px + qy), \quad x, y \in K,$$

and $f(0) = 0$, then f is additive.

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$$F\colon \mathbb{R} \supset [0,+\infty) \to cl(\mathbb{R})$$

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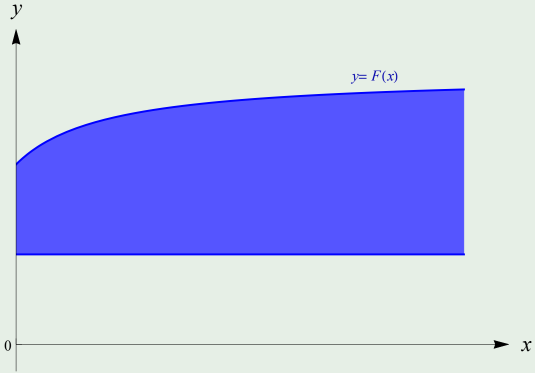

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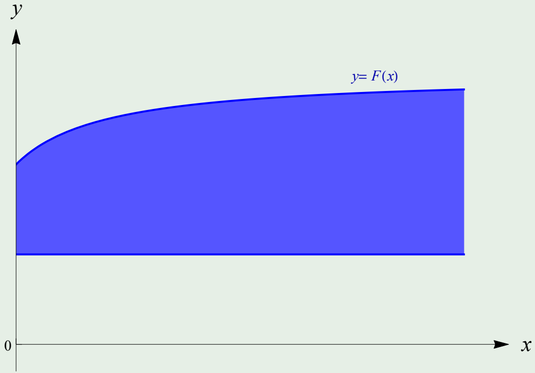

$$\lambda F(x) + (1-\lambda)F(y) \subset F(\lambda x + (1-\lambda)y), \quad x, y \in [0, +\infty), \;\; \lambda \in [0, 1].$$

$$cl(Y) = \{A \subset Y : \ A - \textit{nonempty and closed}\}$$
$$\mathrm{diam} A = \sup\{\|x-y\|, \ x, y \in A\}$$

Theorem 4 (A. Smajdor, J.S.)

Let $\alpha \in (-1, 1)$, $p, q > 0$ and K be a subset of a vector space X such that $0 \in K$ and $K \subset pK$. Assume that $(Y, \|\cdot\|)$ is a real Banach space and $F\colon K \to cl(Y)$ a set-valued function such that

1. $\sup\{\mathrm{diam} F(x), \ x \in K\} = M < +\infty$,
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There exists a unique function $f\colon K \to Y$ such that $f(0) = 0$,

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$$f(x) + F(0) \subset F(x), \ x \in K.$$

Proof. Fix $a \in F(0)$.

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6. $\alpha g_a(x) + (1-\alpha) g_a(y) = g_a(px+qy), \ x, y \in K \ px+qy \in K.$
7. g_a does not depend on the choice of a.

Corollary 5

Let $\alpha \in (-1, 2) \setminus \{0, 1\}$, $p, q > 0$. Assume that K is a convex cone in a vector space X, $(Y, \|\cdot\|)$ a real Banach space and $F : K \to cl(Y)$ a set-valued function with the bounded diameter. If

$$\alpha F(x) + (1 - \alpha)F(y) \subset F(px + qy), \quad x, y \in K,$$

then there exists a unique additive function $f : K \to Y$ such that

$$f(x) + F(0) \subset F(x), \; x \in K.$$

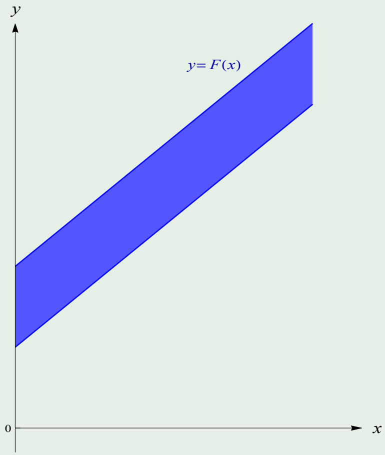
Moreover

$$\alpha f(x) + (1 - \alpha)f(y) = f(px + qy), \quad x, y \in K.$$

Example 6

$$F : \mathbb{R} \supset [0, +\infty) \to cl(\mathbb{R})$$

$$F(x) = [x + 1, x + 2] = x + F(0),\ \ x \in [0, +\infty).$$



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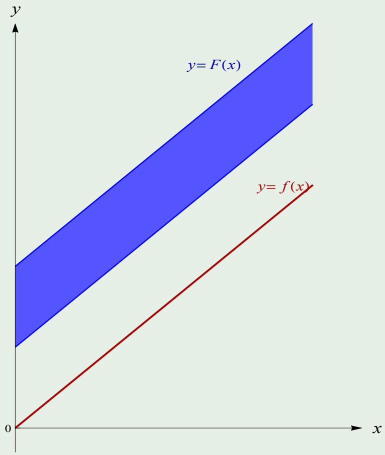

$$ccl(Y) = \{A \subset Y : \ A\text{--}nonempty, \ convex \ and \ closed\}$$

Corollary 7

Let K be a convex cone in a real vector space X, $(Y, \|\cdot\|)$ a real Banach space and $\alpha, \beta, p, q > 0$. Consider a set-valued function $F : K \to ccl(Y)$ such that $\sup\{\mathrm{diam} F(x), \ x \in K\} < +\infty$, *and*

$$\alpha F(x) + \beta F(y) \subset F(px + qy), \quad x, y \in K.$$

1. *If $\alpha + \beta < 1$, then there exists a unique selection $f : K \to Y$ of the multifunction F such that*

$$\alpha f(x) + \beta f(y) = f(px + qy), \ x, y \in K. \tag{1}$$

   Hence f is additive.
2. *If $\alpha + \beta > 1$, then F is single-valued (and it is additive).*
3. *If $\alpha + \beta = 1$, then there exists a unique additive function $f : K \to Y$ satisfying the equation (1) such that*

$$f(x) + F(0) \subset F(x), \ x \in K.$$

Can we instead of condition

$$\sup\{\operatorname{diam} F(x),\ x \in K\} < +\infty$$

assume

$$K \ni x \mapsto \operatorname{diam} F(x) \in \mathbb{R}$$

maps bounded sets onto bounded sets?

Theorem 8

Let K be a convex cone in a normed space $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ a real Banach space and $\alpha, \beta, p, q > 0$. Consider a set-valued function $F\colon K \to ccl(Y)$ such that

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Assume, that

$$p + q \geqslant 1.$$

If $\alpha + \beta < 1$, then there exists a unique selection $f : K \to Y$ of the multifunction F such that

$$\alpha f(x) + \beta f(y) = f(px + qy), \; x, y \in K.$$

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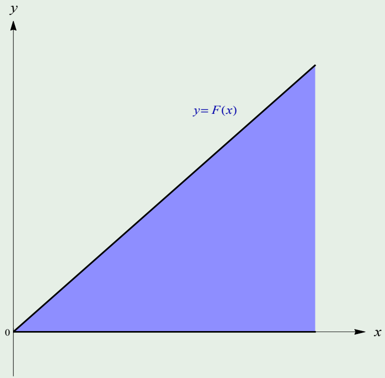

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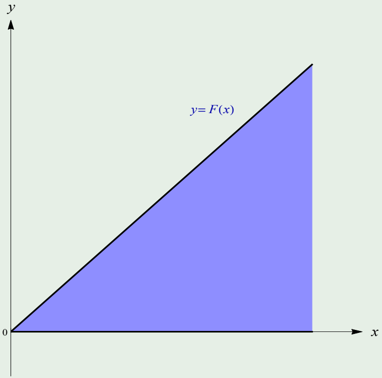

$$\alpha F(x) + \beta F(y) = F(\alpha x + \beta y), \quad x, y \in [0, +\infty), \ \alpha, \beta > 0.$$

$$c(Y) = \{A \subset Y : \ A - \textit{nonempty and compact}\}$$

Theorem 10 (A. Smajdor, J.S.)

Let $p, q > 0$, $\alpha, \beta \in \mathbb{R}$ *Assume that* K *is a convex cone in a normed space* $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ *a real Banach space and* $F\colon K \to c(Y)$ *a set-valued function such that* $0 \in F(0)$,

$$K \ni x \mapsto \operatorname{diam} F(x) \in \mathbb{R}$$

maps bounded sets onto bounded sets and

$$\alpha F(x) + \beta F(y) \subset F(px + qy), \quad x, y \in K.$$

If

$$|\alpha| < p \quad (\textit{or } |\beta| < q),$$

then there exists a unique selection $f : K \to Y$ *of* F *fulfilling the equation*

$$\alpha f(x) + \beta f(y) = f(px + qy), \quad x, y \in K.$$

The selection f *is additive.*

Theorem 11 (A. Smajdor, J.S.)

Let $\alpha \in (-1,1)$, $p, q > 0$, K be a convex cone in a normed space $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ a real Banach space. Assume that $F\colon K \to bcl(Y)$ is a set-valued function such that

$$K \ni x \ \mapsto \ \operatorname{diam} F(x) \in [0, +\infty)$$

maps bounded subsets of K onto bounded and

$$\alpha F(x) + (1-\alpha)F(y) \subset F(px + qy), \quad x, y \in K.$$

If $p \geqslant 1$, then there exists a unique function $f\colon K \to Y$ such that

$$f(x) + F(0) \subset F(x), \; x \in K$$

and fulfilling the functional equation

$$\alpha f(x) + (1-\alpha)f(y) = f(px + qy), \quad x, y \in K.$$

Moreover, f is an additive function.