The cardinality of subgroups characterized by a pointwise convergence of characters

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28th International Summer Conference on Real Functions Theory
Stará Lesná, 2014

Notation

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the circle group.

Let $\|x\|$ denote the distance of $x \in \mathbb{T}$ to the zero element.

A subgroup G of the unit circle $\mathbb{T}$ is characterized by a pointwise convergence of characters if there exists an increasing sequence of integers $\{a_n\}_{n\in\mathbb{N}}$ such that $G = \{x \in \mathbb{T} : \lim_{n\to\infty} \|a_n x\| = 0\}$.

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For which sequences $\{n_k\}_{k\in\mathbb{N}}$ is the group G uncountable?

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For $\boldsymbol{a} = \{a_n\}_{n\in\mathbb{N}}$, an increasing sequence of positive integers, denote $A(\boldsymbol{a}) = \{x \in \mathbb{T} : \lim_{n\to\infty} \|a_n x\| = 0\}$.

$A(\boldsymbol{a})$ is a proper $\boldsymbol{\Pi}^0_3$ subgroup of $\mathbb{T}$.

(Closed) subsets of these groups are called Arbault sets.

Theorem (Folklore)

1. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 0$ *then* $A(\boldsymbol{a})$ *is uncountable.*

2. If $\displaystyle\liminf_{n\to\infty} \frac{a_n}{a_{n+1}} > 0$ *then* $A(\boldsymbol{a})$ *is countable.*

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Expressing integers as combinations of integers

Let $\boldsymbol{a} = \{a_n\}_{n\in\mathbb{N}}$, $\boldsymbol{z} = \{z_n\}_{n\in\mathbb{N}}$ be sequences of integers, $m \in \mathbb{Z}$.
$\boldsymbol{z}$ is called an expansion of m by $\boldsymbol{a}$ if $m = \sum_{n\in\mathbb{N}} z_n a_n$.

It is called a good expansion if, for every n, $\left|\sum_{j<n} z_j a_j\right| \leq \frac{a_n}{2}$.

Lemma

If $\boldsymbol{a} = \{a_n\}_{n\in\mathbb{N}}$ is increasing, $a_0 = 1$ and $m \in \mathbb{Z}$, then there exists a good expansion of m by $\boldsymbol{a}$.

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Inclusions between Arbault sets

Let $\boldsymbol{a} = \{a_n\}_{n\in\mathbb{N}}$ and $\boldsymbol{b} = \{b_k\}_{k\in\mathbb{N}}$ be increasing sequences of positive integers, $a(0) = 1$, and $\lim\limits_{n\to\infty} \frac{a_n}{a_{n+1}} = 0$.

For every $k \in \mathbb{N}$, let $\{z_{k,n}\}_{n\in\mathbb{N}}$ be a good expansion of b_k by $\boldsymbol{a}$.

Theorem

$A(\boldsymbol{a}) \subseteq A(\boldsymbol{b})$ holds true if and only if

1. $\forall n \ \exists K \ \forall k \geq K \ z_{k,n} = 0$,
2. $\exists M \ \forall k \ \sum_n |z_{k,n}| \leq M$.

Problem. Is assumption $\lim\limits_{n\to\infty} \frac{a_n}{a_{n+1}} = 0$ necessary?

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Easy part: If 1. and 2. hold true then $A(\boldsymbol{a}) \subseteq A(\boldsymbol{b})$.

Uneasy part: If 2. does not hold then either

A. $\{|z_{k,n}| : k, n \in \mathbb{N}\}$ is unbounded, or

B. $\{|\{n : z_{k,n} \neq 0\}| : k \in \mathbb{N}\}$ is unbounded.

If 2. holds true and 1. does not hold then

C. there is n and an infinite set J such that for every $k \in J$, $z_{k,n} \neq 0$ and $\{m > n : z_{k,m} \neq 0\}$ is finite.

In each case we find $x \in A(\boldsymbol{a}) \setminus A(\boldsymbol{b})$.

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A sufficient condition

Corollary

Let $\boldsymbol{b} = \{b_k\}_{k\in\mathbb{N}}$ be an increasing sequence of positive integers. If there exist sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{z_{k,n}\}_{k,n\in\mathbb{N}}$ such that

1. $\lim\limits_{n\to\infty} \dfrac{a_n}{a_{n+1}} = 0$,
2. $\forall n\ \exists K\ \forall k \geq K\ \ z_{k,n} = 0$,
3. $\exists M\ \forall k\ \ \sum_n |z_{k,n}| \leq M$,
4. $\forall k\ b_k = \sum_n z_{k,n} a_n$,

then $A(\boldsymbol{b})$ is uncountable.

Questions. Which sequences b have this property?
Is this condition necessary?

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then $A(\boldsymbol{b})$ *is uncountable.*

Questions. Which sequences $\boldsymbol{b}$ have this property?
Is this condition necessary?

Answer to the second question

Let $\{c_k\}_{k\in\mathbb{N}}$, $\{m_k\}_{k\in\mathbb{N}}$ be sequences of positive integers such that

- $\{m_k\}_{k\in\mathbb{N}}$ is unbounded and lists each value infinitely many times, and
- $\lim_{k\to\infty} \frac{m_k c_k}{c_{k+1}} = 0.$

Let $b_{2k} = c_k$, $b_{2k+1} = m_k c_k$.

Lemma

1. $\boldsymbol{b} = \{b_k\}_{k\in\mathbb{N}}$ does not satisfy assumptions of the corollary.
2. $A(\boldsymbol{b})$ is uncountable.

In fact, $A(\boldsymbol{b})$ does not contain any subgroup of the form $A(\boldsymbol{a})$ where $\boldsymbol{a} = \{a_n\}_{n\in\mathbb{N}}$ satisfies $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 0$.

It does contain uncountable set $\{x \in \mathbb{T} : \forall k \ \ \|c_k x\| \le c_k/c_{k+1}\}$.

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Dirichlet groups

For $\boldsymbol{a} = \{a_n\}_{n\in\mathbb{N}}$ an increasing sequence of positive integers, and $\boldsymbol{\varepsilon} = \{\varepsilon_n\}_{n\in\mathbb{N}}$ a sequence of positive reals, denote $D(\boldsymbol{a}, \boldsymbol{\varepsilon}) = \{x \in \mathbb{T} : \forall n \ \ \|a_n x\| \le \varepsilon_n\}$.

If $\lim_{n\to\infty} \varepsilon_n = 0$ then the group generated by $D(\boldsymbol{a}, \boldsymbol{\varepsilon})$ is a proper Σ_2^0 subgroup of $\mathbb{T}$.

Groups of this form are called Dirichlet groups, subsets of $D(\boldsymbol{a}, \boldsymbol{\varepsilon})$ are Dirichlet sets.

Lemma

1. *If for every n, $\varepsilon_n \ge \dfrac{a_n}{a_{n+1}}$, then $D(\boldsymbol{a}, \boldsymbol{\varepsilon})$ is uncountable.*
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