On some modification of density zero ideal

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joint work with Marek Balcerzak, Pratulananda Das
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Background

The lower and the upper densities of $A \subset \omega$ are given by the formulas

$$\underline{d}(A) = \liminf_{n \to \infty} \frac{card(A \cap n)}{n}$$

$$\overline{d}(A) = \limsup_{n \to \infty} \frac{card(A \cap n)}{n}.$$

If $\underline{d}(A) = \overline{d}(A)$, we say that the natural density of A exists and it is denoted by $d(A)$.

We say that a sequence (x_n) of real numbers is statistically convergent to $g \in \mathbb{R}$ if for any $\epsilon > 0$ we have

$$d(\{n \in \omega : |g - x_n| > \epsilon\}) = 0.$$

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Let X be any set. A family $\mathcal{I} \subset \mathcal{P}(X)$ is called an ideal on X whenever

- $\emptyset \in J$ and $X \notin J$,
- if $A, B \in J$ then $A \cup B \in J$,
- $A \subset B$ and $B \in J$ then $A \in J$.

We will consider ideals on ω. In this case it is natural to assume that considered ideals contains Fin (ideal of finite subsets of ω).

An ideal $\mathcal{I}$ on ω is called a P-ideal if for every sequence $(A_n)_{n\in\omega}$ of sets in $\mathcal{I}$ there is a set $A \in \mathcal{I}$ such that $A_n \subset^* A$ for all $n \in \omega$ (where $A_n \subset^* A$ means that $A_n \setminus A \in Fin$).

Every ideal $\mathcal{I}$ on ω can be treated as a subset of the Cantor space 2^ω since $P(\omega)$ and 2^ω can be identified via the characteristic functions.

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A submeasure on ω is a function $\varphi\colon \mathcal{P}(\omega) \to [0,\infty]$ such that:

- $\varphi(\emptyset) = 0$;
- if $A \subset B$ then $\varphi(A) \leq \varphi(B)$,
- $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$,
- $\varphi(\{n\}) < \infty$ for all $n \in \omega$.

A submeasure φ is called a lower semicontinuos submeasure (in short, lscsm) if $\varphi(A) = \lim_{n\to\infty} \varphi(A \cap n)$ for all $A \subset \omega$. For any lscsm φ, we consider two ideals given by

$$Exh(\varphi) = \{A \subset \omega\colon \lim_{n\to\infty} \varphi(A \setminus n) = 0\}.$$

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Let φ be a lscsm. Then $Exh(\varphi)$ is an $F_{\sigma\delta}$ P-ideal, $Fin(\varphi)$ is an F_σ ideal and $Exh(\varphi) \subset Fin(\varphi)$.

Theorem [Mazur, Solecki]

Let $\mathcal{I}$ be an ideal on ω. Then

- $\mathcal{I}$ is an F_σ ideal if and only if $\mathcal{I} = Fin(\varphi)$ for some lscsm φ.
- $\mathcal{I}$ is an analytic P-ideal if and only if $\mathcal{I} = Exh(\varphi)$ for some lscsm φ.
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We define a density ideal $\mathcal{I}_d$ as $\mathcal{I}_d := \{A \subset \omega : d(A) = 0\}$. It may be shown that $\mathcal{I}_d = Exh(\varphi)$ for lscsm φ given by

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This idea can go further. We fix any function $g\colon \omega \to [0,\infty)$ with $\lim_{n\to\infty} g(n) = \infty$. Then we define the upper density of weight g by the formula

$$\overline{d_g}(A) = \limsup_{n\to\infty} \frac{card(A \cap n)}{g(n)} \quad \text{for } A \subset \omega$$

Consider the following family

$$\mathbb{Z}_g := \{A \subset \omega \colon \overline{d_g}(A) = 0\}.$$

Of course $\omega \in \mathbb{Z}_g \iff n/g(n) \to 0$. So, if we additionally assume $n/g(n) \nrightarrow 0$ then $\omega \notin \mathbb{Z}_g$, and we observe that $\mathbb{Z}_g$ is an ideal on ω. Note that $Fin \subsetneq \mathbb{Z}_g$.

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Results

Proposition

If $g : \omega \to [0, \infty)$ is such that $g(n) \to \infty$ and $n/g(n) \nrightarrow 0$, then the ideal $\mathbb{Z}_g$ is equal to $Exh(\varphi)$ where

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Let $g_1, g_2 \in G$ be such that there exist $M > 0$ and $k \in \omega$ such that $g_1(n)/g_2(n) \leq M$ for all $n \geq k$. Then $\mathbb{Z}_{g_1} \subset \mathbb{Z}_{g_2}$. Consequently, if there exist $0 < m < M$ and $k \in \omega$ such that $m \leq g_1(n)/g_2(n) \leq M$ for all $n \geq k$, then $\mathbb{Z}_{g_1} = \mathbb{Z}_{g_2}$.

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Lemma

Let $f \in G$ be such that $n/f(n) \to \infty$. Then there exists a set $A \subset \omega$ such that the sequence $(card(A \cap n)/f(n))$ is bounded but not convergent to 0.

Theorem

If $g_1, g_2 \in G$ are such that $n/g_2(n) \to \infty$, $g_2(n)/g_1(n) \to \infty$ then $\mathbb{Z}_{g_1} \subsetneq \mathbb{Z}_{g_2}$. If $g \in G$ and $n/g(n) \to \infty$ then $\mathbb{Z}_g \subsetneq \mathbb{Z}$.

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There exists a function $g \in G$ such that $\mathcal{Z}_g \subsetneq \mathcal{Z}$ and $\mathcal{Z}_g$ is different from any ideal generated by a function of the form n^α with $0 < \alpha < 1$.

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Theorem

There exists a family $G_0 \subset G$ of cardinality $\mathfrak{c}$ such that $\mathbb{Z}_f$ is incomparable with $\mathbb{Z}$ for every $f \in G$, and $\mathbb{Z}_f$ and $\mathbb{Z}_g$ are incomparable for any distinct $f, g \in G_0$.

Examples

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Thank you for your attention!

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