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0-dimensional uncountable compact spaces as attractors of iterated function systems

(with Taras Banakh and Magdalena Nowak)

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Let (X, d) be a metric space.

* By $\mathcal{K}(X)$ we denote the space of all nonempty and compact subsets of X.
* We consider $\mathcal{K}(X)$ as a metric space with the Hausdorff metric H:

$$H(K, D) := \max\left\{\sup_{x \in K}\left(\inf_{y \in D} d(x, y)\right), \sup_{y \in D}\left(\inf_{x \in K} d(x, y)\right)\right\}$$

* If $\mathcal{S} = (f_1, ..., f_n)$ is a finite system of continuous selfmaps of X (i.e., $f_i : X \to X$), then by $F_{\mathcal{S}}$ we denote the function $F_{\mathcal{S}} : \mathcal{K}(X) \to \mathcal{K}(X)$ defined by

$$F_{\mathcal{S}}(K) := f_1(K) \cup ... \cup f_n(K)$$

* If $f : X \to X$, then by $Lip(f)$ we denote the Lipschitz constant of f.
* If $Lip(f) < 1$, that is there exists $\alpha < 1$ such that

$$\forall_{x,y \in X}\ d(f(x), f(y)) \le \alpha d(x, y),$$

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IFSs and their attractors

Theorem, Hutchinson, Barnsley, 1980's

Let X be a complete metric space and $\mathcal{S} = (f_1, ..., f_n)$ be a finite system of Banach contractions of X. Then there exists a unique $A_\mathcal{S} \in \mathcal{K}(X)$ such that

$$A_\mathcal{S} = F_\mathcal{S}(A_\mathcal{S}) = f_1(A_\mathcal{S}) \cup ... \cup f_n(A_\mathcal{S})$$

Moreover, for every $K \in \mathcal{K}(X)$, the sequence of iterates $F_\mathcal{S}^{(k)}(K) \to A_\mathcal{S}$ with respect to the Hausdorff metric.

Remark

In the above result it is enough to assume that each f_i is a weak contraction (Rakotch, Browder, Matkowski).

Fractals in the sense of Hutchinson and Barnsley

* If a function system $\mathcal{S}$ satisfies the thesis of the above theorem, then we say that $\mathcal{S}$ *generates a fractal*.
* A compact set $A_\mathcal{S}$ which is generated by a function system $\mathcal{S}$ in this way is called *fractal* or *attractor in the sense of Hutchinson and Barnsley*.
* In this setting, function systems are called *iterated function systems* (IFSs).

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Major questions concerning Hutchinson-Barnsley theory

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(Q1) Which iterated function systems generate fractals?

(Q2) What is the structure of fractals generated by IFSs?

(Q3) Which compact spaces are fractals generated by IFSs?

Variants of (Q3)

(3a) Which compact meric spaces are fractals generated by IFSs consisting of Banach contractions?

(3b) Which compact metric spaces are fractals generated by IFSs consisting of weak contractions?

(3c) Which compact metric spaces are homeomorphic to fractals generated by IFSs?

(3c') Which compact metric spaces X have the following property:

There is an IFS $\mathcal{S} = (f_1, ..., f_n)$ such that $X = F_{\mathcal{S}}(X) = f_1(X) \cup ... \cup f_n(X)$, and an admissible metric d on X so that each f_i is a weak (or Banach) contraction with respect to d.

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Major questions and known results

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Which compact spaces are fractals generated by IFSs?

Known results

* (Crovisier and Rams [CR] 2006, S. [S] 2013): There is a Cantor set $C \subset \mathbb{R}$ which is not an attractor of any IFS consisting of weak contractions.
* (Sanders [Sa] 2002) An arc $A \subset \mathbb{R}^n$ is the attractor of an IFS consisting of Banach contractions on $\mathbb{R}^n$ iff its variation $V(A) < \infty$.
* (Banakh, Nowak [BN] 2013): There is an 1-dimensional Peano continuum which not homeomorphic to the attractor of any IFS consisting of Banach contractions.
* (Nowak [N] 2013): For a countable and compact space X, TFCAE:
  - (i) X is homeomorphic to the attractor of a certain IFS consisting of Banach contractions;
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  - (iii) the scattered height of X is a successor ordinal.

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Main result

Theorem 1, Banakh, Nowak, S., 2014

Let X be an uncountable, 0-dimensional compact space. Then there is an IFS $\mathcal{S} = (f, g)$ such that $X = F_{\mathcal{S}}(X) = f(X) \cup g(X)$, and for every $c \in (0, 1)$ there is an admissible metric d on X such that $Lip(f), Lip(g) \leq c$.

Remark

Recall that there is a Cantor set on $\mathbb{R}$ which is not the attractor of an IFS on consisting of weak contractions.

Corollary

If X is an uncountable compact metrizable space, then TFCAE:

(i) X is 0-dimensional;

(ii) X satisfies the thesis of Theorem 1;

(iii) X is homeomorphic to an IFS $\mathcal{S} = (f_1, ..., f_n)$ with $\sum_{i=1}^{n} Lip(f_i) < 1$.

Theorem, Daniello, Steele, 2014

Each compact uncountable nowhere dense subset of $\mathbb{R}$ is homeomorphic to the attractor of an IFS consisting of Banach contractions.

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Thank you for your attention

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