On Haar meager sets

Eliza Jabłońska

Department of Mathematics
Rzeszów University of Technology,
Rzeszów, Poland

The 28th International Summer Conference
on Real Functions Theory
Stara Lesna, August 31 - September 5, 2014

1972 J.P.R. Christensen [1] defined *Haar null* sets in an abelian Polish group (a topological abelian group with a complete separable metric) in such a way that in a locally compact group it is equivalent to the notation of Haar measure zero sets.

1972 J.P.R. Christensen [1] defined *Haar null* sets in an abelian Polish group (a topological abelian group with a complete separable metric) in such a way that in a locally compact group it is equivalent to the notation of Haar measure zero sets.

Definition (Christensen [1])

Let X be an abelian Polish group. A set $A \subset X$ is *Haar null* if there is a universally measurable set $B \subset X$ with $A \subset B$ and a probability measure ν on X (not unique) such that $\nu(B + x) = 0$ for all $x \in X$.

1972 J.P.R. Christensen [1] defined *Haar null* sets in an abelian Polish group (a topological abelian group with a complete separable metric) in such a way that in a locally compact group it is equivalent to the notation of Haar measure zero sets.

Definition (Christensen [1])

Let X be an abelian Polish group. A set $A \subset X$ is *Haar null* if there is a universally measurable set $B \subset X$ with $A \subset B$ and a probability measure ν on X (not unique) such that $\nu(B + x) = 0$ for all $x \in X$.

1992 J. Mycielski [9] extended this definition to non-abelian groups.

1972 J.P.R. Christensen [1] defined *Haar null* sets in an abelian Polish group (a topological abelian group with a complete separable metric) in such a way that in a locally compact group it is equivalent to the notation of Haar measure zero sets.

Definition (Christensen [1])

Let X be an abelian Polish group. A set $A \subset X$ is *Haar null* if there is a universally measurable set $B \subset X$ with $A \subset B$ and a probability measure ν on X (not unique) such that $\nu(B + x) = 0$ for all $x \in X$.

1992 J. Mycielski [9] extended this definition to non-abelian groups.

1992 B.R. Hunt, T. Sauer and J.A. Yorke [4]-[5] (unaware of Christensen's result) found this notation again, but in a topological abelian group with a complete metric (not necessary separable).

2013 U.B. Darji [2] introduced another family of "small" sets in an abelian Polish group, which is equivalent to the notation of meager sets in a locally compact group.

2013 U.B. Darji [2] introduced another family of "small" sets in an abelian Polish group, which is equivalent to the notation of meager sets in a locally compact group.

Definition (Darji [2])

Let X be an abelian Polish group. A set $A \subset X$ is *Haar meager* if there is a Borel set $B \subset X$ with $A \subset B$, a compact metric space K and a continuous function $f : K \to X$ such that $f^{-1}(B + x)$ is meager in K for all $x \in X$.

2013 U.B. Darji [2] introduced another family of "small" sets in an abelian Polish group, which is equivalent to the notation of meager sets in a locally compact group.

Definition (Darji [2])

Let X be an abelian Polish group. A set $A \subset X$ is *Haar meager* if there is a Borel set $B \subset X$ with $A \subset B$, a compact metric space K and a continuous function $f : K \to X$ such that $f^{-1}(B + x)$ is meager in K for all $x \in X$.

$$\mathcal{HM} := \{A \subset X : A \text{ is Haar meager}\}$$
$$\mathcal{M} := \{A \subset X : A \text{ is meager}\}.$$

Theorem (Darji [2])

Let X be an abelian Polish group. The family $\mathcal{HM}$ is a σ-ideal such that $\mathcal{HM} \subset \mathcal{M}$.

Theorem (Darji [2])

Let X be an abelian Polish group. The family $\mathcal{HM}$ is a σ-ideal such that $\mathcal{HM} \subset \mathcal{M}$.

Theorem (Darji [2], Solecki [10], Matoušková, Zelený [7])

- *Let X be a locally compact abelian Polish group. Then $\mathcal{M} \subset \mathcal{HM}$ and consequently $\mathcal{M} = \mathcal{HM}$.*
- *Let X be a non-locally compact abelian Polish group. Then $\mathcal{M} \not\subset \mathcal{HM}$; more precisely, there is a closed nowhere dense set $A \notin \mathcal{HM}$.*

Theorem (Christensen [1])

Let X be an abelian Polish group. Then $0 \in int(A - A)$ for each non-Haar null universally measurable set $A \subset X$.

Theorem (Christensen [1])

Let X be an abelian Polish group. Then $0 \in int(A - A)$ for each non-Haar null universally measurable set $A \subset X$.

Theorem (J [6])

Let X be an abelian Polish group. If $A \subset X$ is a Borel non-Haar meager set, then $0 \in int(A - A)$.

Sketch of the proof

- Suppose that $0 \notin \operatorname{int}(A - A)$;
- $F(A) := \{x \in X : (x + A) \cap A \notin \mathcal{HM}\}$;
- $0 \in F(A) \subset A - A$;
- choose $(x_i)_{i \in \mathbb{N}} \subset X \setminus F(A)$ such that

$$\rho(0, x_i) \leq \frac{1}{2^i} \quad \text{for each} \quad i \in \mathbb{N};$$

- $A_0 := A \setminus \left[\bigcup_{i \in \mathbb{N}} (x_i + A) \cap A\right]$;
- $A_0 \notin \mathcal{HM}$, i.e. for every compact metric space K and continuous function $g : K \to X$ there is a $y_K \in X$ such that $g^{-1}(A_0 + y_K)$ is non-meager in K;

Sketch of the proof

- $K := \{0,1\}^{\aleph_0}$ is a a compact metric group with $\oplus : K \times K \to K$ given by

$$(k_i)_{i\in\mathbb{N}} \oplus (l_i)_{i\in\mathbb{N}} := (k_i +_2 l_i)_{i\in\mathbb{N}} \text{ for every } (k_i)_{i\in\mathbb{N}}, (l_i)_{i\in\mathbb{N}} \in K$$

  ($+_2$ is the operation mod 2) and with the product metric

$$d((k_i)_{i\in\mathbb{N}}, (l_i)_{i\in\mathbb{N}}) := \sum_{i=1}^{\infty} 2^{-i}\overline{d}(k_i, l_i)$$

  for every $(k_i)_{i\in\mathbb{N}}, (l_i)_{i\in\mathbb{N}} \in K$ ($\overline{d}$ is the discrete metric in $\{0,1\}$);
- $g : K \to X$ is well defined and uniformly continuous on K

$$g((k_i)_{i\in\mathbb{N}}) := \sum_{i=1}^{\infty} k_i x_i \text{ for } (k_i)_{i\in\mathbb{N}} \in K;$$

Sketch of the proof

- the set $g^{-1}(A_0 + y_K)$ is non-meager with the Baire property in K for some $y_K \in X$;
- by Piccard's theorem, there is an open ball $K(0, 2^{-k})$ with some $k \in \mathbb{N}$ such that

$$K(0, 2^{-k}) \subset g^{-1}(A_0 + y_K) - g^{-1}(A_0 + y_K);$$

- $e_{k+1} := (0, \ldots, 0, \underbrace{1}_{k+1}, 0, 0, \ldots) \in K(0, 2^{-k})$;
- $e_{k+1} = a - b$ for some $a, b \in g^{-1}(A_0 + y_K)$;
- $\pm x_{k+1} = g(a) - g(b) \in A_0 - A_0$;
- $(x_{k+1} + A_0) \cap A_0 \neq \emptyset$, what contradicts the definition of A_0.

Corollary

In a non-locally compact abelian Polish group each compact set is Haar meager as well as Haar null.

Corollary

In a non-locally compact abelian Polish group each compact set is Haar meager as well as Haar null.

Fact

Let X be an abelian Polish group and

$$\mathcal{F}(X) := \{A \subset X : \forall_{K \subset X\text{-compact}} \ \exists_{x_K \in X} \ K + x_K \subset A\}.$$

Each Borel set $A \in \mathcal{F}(X)$ is neither Haar null, nor Haar meager.

Corollary

In a non-locally compact abelian Polish group each compact set is Haar meager as well as Haar null.

Fact

Let X be an abelian Polish group and

$$\mathcal{F}(X) := \{A \subset X : \forall_{K \subset X\text{-compact}} \; \exists_{x_K \in X} \; K + x_K \subset A\}.$$

Each Borel set $A \in \mathcal{F}(X)$ is neither Haar null, nor Haar meager.

S. Solecki [10] and also E. Matoušková, M. Zelený [7] showed how to find a closed nowhere dense set from the family $\mathcal{F}(X)$ in any abelian non-locally compact Polish group X.

Corollary

In a non-locally compact abelian Polish group each compact set is Haar meager as well as Haar null.

Fact

Let X be an abelian Polish group and

$$\mathcal{F}(X) := \{A \subset X : \forall_{K \subset X\text{-compact}} \ \exists_{x_K \in X} \ K + x_K \subset A\}.$$

Each Borel set $A \in \mathcal{F}(X)$ is neither Haar null, nor Haar meager.

S. Solecki [10] and also E. Matoušková, M. Zelený [7] showed how to find a closed nowhere dense set from the family $\mathcal{F}(X)$ in any abelian non-locally compact Polish group X.

Problem

What about Haar meager non-Haar null sets, or Haar null non-Haar meager sets?

S. Solecki [10] and also E. Matoušková, M. Zelený [7] showed how to find a Borel set from the family $\mathcal{F}(X)$ in any abelian non-locally compact Polish group X. We will use this fact to construct a Haar meager non-Haar null set, as well as a meager Haar null set which is not Haar meager in spaces of sequences: l_p with $p \geq 1$, c_0 or c. Such spaces have a very nice property: $X = \mathbb{R} \times X$.

S. Solecki [10] and also E. Matoušková, M. Zelený [7] showed how to find a Borel set from the family $\mathcal{F}(X)$ in any abelian non-locally compact Polish group X. We will use this fact to construct a Haar meager non-Haar null set, as well as a meager Haar null set which is not Haar meager in spaces of sequences: l_p with $p \geq 1$, c_0 or c. Such spaces have a very nice property: $X = \mathbb{R} \times X$.

Theorem (J [6])

Let X, Y be abelian Polish groups.

- *If $A \subset X$ is Haar meager and $B \subset B_0$ for some Borel set $B_0 \subset Y$, then the set $A \times B \subset X \times Y$ is Haar meager.*
- *For every Borel set $A \in \mathcal{F}(X)$ and non-Haar meager Borel set $B \subset Y$ the set $A \times B \subset X \times Y$ is not Haar meager.*

The above theorem suggest the following

Problem

Let X, Y be abelian Polish groups. Is it true or false that for every non-Haar meager Borel sets $A \subset X$ and $B \subset Y$ the set $A \times B \subset X \times Y$ is not Haar meager?

The above theorem suggest the following

Problem

Let X, Y be abelian Polish groups. Is it true or false that for every non-Haar meager Borel sets $A \subset X$ and $B \subset Y$ the set $A \times B \subset X \times Y$ is not Haar meager?

A negative answer implies the same question under additional assumption that one of abelian Polish groups X, Y is locally compact. In fact the above problem is tightly connected with the following question:

Problem

Can the Kuratowski-Ulam theorem be generalized on the case of Haar meager sets?

Example (Christensen [1])

If $X = H$ is a separable infinite dimensional Hilbert space and $Y = T$ is the unit circle in the complex plane, there is a Borel Haar null set $A \subset X \times Y$ such that sections $A[y]$ are not Haar null sets in X for every $y \in Y$.

Example (Christensen [1])

If $X = H$ is a separable infinite dimensional Hilbert space and $Y = T$ is the unit circle in the complex plane, there is a Borel Haar null set $A \subset X \times Y$ such that sections $A[y]$ are not Haar null sets in X for every $y \in Y$.

Theorem (Christensen [1])

If X is an abelian Polish group, Y is a locally compact abelian Polish group, $A \subset X \times Y$ is a universally measurable set and sections $A[x]$ are sets of Haar measure zero in Y for almost every $x \in X$, then A is Haar null in $X \times Y$.

Now, consider the space X as one of the following spaces of sequences: c_0, c or l_p with $p \geq 1$. Next, fix a Borel meager set $A \in \mathcal{F}(X)$.

Now, consider the space X as one of the following spaces of sequences: c_0, c or l_p with $p \geq 1$. Next, fix a Borel meager set $A \in \mathcal{F}(X)$.

Meager, Haar null, non-Haar meager set

Let $S := B \times A \subset \mathbb{R} \times X$, where $B \subset \mathbb{R}$ is a non-meager set of the Lebesgue measure zero. By Christensen's theorem S is Haar null in X. Moreover, the set S is not Haar meager in $\mathbb{R} \times X = X$. Finally, S is meager according to the Kuratowski-Ulam theorem, because the set A is meager.

Haar meager, non-Haar null set

Let $T := C \times A \subset \mathbb{R} \times X$, where $C \subset \mathbb{R}$ is a meager set of the positive Lebesgue measure such that $\mathbb{R} \setminus C$ is a set of the Lebesgue measure zero. By Christensen's theorem $(\mathbb{R} \setminus C) \times A$ is Haar null. Moreover, $\mathbb{R} \times A$ is not Haar null.
To show it contrary suppose that $\mathbb{R} \times A$ is Haar null, i.e. there is a probability Borel measure μ on $\mathbb{R} \times X$ such that $\mu((\mathbb{R} \times A) + (r, x)) = 0$ for each $(r, x) \in \mathbb{R} \times X$. Define $\nu(M) = \mu(\mathbb{R} \times M)$ for each Borel set $M \subset X$. Then ν is a probability Borel measure on X and, for each $x \in X$ we have

$$\nu(A + x) = \mu(\mathbb{R} \times (A + x)) = \mu((\mathbb{R} \times A) + (0, x)) = 0.$$

It means that A is Haar null, what is a contradiction.
Since $\mathbb{R} \times A$ is not Haar null, $T = C \times A = (\mathbb{R} \times A) \setminus ((\mathbb{R} \setminus C) \times A)$ is not Haar null, too. Moreover, the set T is Haar meager.

Example (E. Matoušková, L. Zajíček [8])

Let $A := \{(x_n)_{n\in\mathbb{N}} \in c_0 : \ \forall_{n\in\mathbb{N}}\, x_n \geq 0\}$. Such set is closed, nowhere dense and belongs to the filter $\mathcal{F}(c_0)$. Now, define sets

$$S := B \times A \subset c_0 \quad \text{and} \quad T := C \times A \subset c_0,$$

where $B \subset \mathbb{R}$ is a non-meager set of the Lebesgue measure zero and $C \subset \mathbb{R}$ is a meager set of the positive Lebesgue measure such that $\mathbb{R} \setminus C$ has Lebesgue measure zero. Then the set T is Haar meager, but not Haar null in c_0 and the set S is Haar null, meager, but not Haar meager in c_0.

Problem

Let X be any abelian Polish group, which is not locally compact. How to find a Haar meager non-Haar null set and, conversely, how to construct a Haar null, meager but not Haar meager set in X?

Problem

Let X be any abelian Polish group, which is not locally compact. How to find a Haar meager non-Haar null set and, conversely, how to construct a Haar null, meager but not Haar meager set in X?

Problem

Let X be any abelian Polish group, which is not locally compact. Can X be decomposed on two disjoint sets A and B, where A is Haar meager in X and B is Haar null in X?

If X is the space of sequences, the answer is "yes". Indeed, $\mathbb{R} = S \cup T$, where S is meager in $\mathbb{R}$, T has Lebesgue measure zero in $\mathbb{R}$ and $S \cap T = \emptyset$. Then

$$X = \mathbb{R} \times X = (S \cup T) \times X = (S \times X) \cup (T \times X),$$

where $S \times X$ is Haar meager and $T \times X$ is Haar null.

- J.P.R. Christensen, *On sets of Haar measure zero in abelian Polish groups*, Israel J. Math. **13** (1972), 255-260.
- U.B. Darji, *On Haar meager sets*, Topology Appl. **160** (2013), 2396-2400.
- P. Fischer, Z. Słodkowski, *Christensen zero sets and measurable convex functions*, Proc. Amer. Math.Soc. **79** (1980), 449-453.
- B.R. Hunt, T. Sauer, J.A. Yorke, *Prevalence: a translation-invariant "almost every" on infinite-dimensional spaces*, Bull. Amer. Math. Soc. **27** (1992), 217-238.
- B.R. Hunt, T. Sauer, J.A. Yorke, *Prevalence: an addendum*, Bull. Amer. Math. Soc. **28** (1993), 306-307.

E. Jabłońska, *Some analogies between Haar meager sets and Haar null sets in abelian Polish groups*, J. Math. Anal. Appl. (2014), http://dx.doi.org/10.1016/j.jmaa.2014.08.005.

E. Matoušková, M. Zelený, *A note on intersections of non-Haar null sets*, Colloq. Math. **96** (2003), 1-4.

E. Matoušková, L. Zajíček, *Second order differentiability and Lipschitz smooth points ofconvex functionals*, Czechoslovak Math. J. **48** (1998), 617-640.

J. Mycielski, *Unsolved problems on the prevalence of ergodicity, instability and algebraic independence*, Ulam Quarterly **1** (1992), 30-37.

S. Solecki, *On Haar null sets*, Fund. Math. **149** (1996), 205-210.