Strong measure zero in separable metric spaces and Polish groups

Ondřej Zindulka (with Michael Hrušák and Wolfgang Wohofsky)

Czech Technical University Prague

Stará Lesná 2014

Miller–Steprāns paper

Framework

$\mathbb{G}$ . . . non-discrete Polish group
X . . . Polish space
$\mathcal{M}$ . . . the ideal of meager sets in X
α . . . action of $\mathbb{G}$ on a Polish space X

Definition (Miller–Steprāns 2006)

- $\mathrm{cov}_{\mathbb{G}} = \min\{|A| : \exists M \in \mathcal{M}\ A + M = \mathbb{G}\}$
- $\mathrm{cov}_{\alpha} = \min\{|A| : A \subseteq \mathbb{G}, \exists M \in \mathcal{M}\ \alpha(A \times M) = X\}$

Miller–Steprāns paper

Framework

$\mathbb{G}$... non-discrete Polish group
X... Polish space
$\mathcal{M}$... the ideal of meager sets in X
α... action of $\mathbb{G}$ on a Polish space X

Definition (Miller–Steprāns 2006)

- $\mathsf{cov}_{\mathbb{G}} = \min\{|A| : \exists M \in \mathcal{M}\ A + M = \mathbb{G}\}$
- $\mathsf{cov}_{\alpha} = \min\{|A| : A \subseteq \mathbb{G}, \exists M \in \mathcal{M}\ \alpha(A \times M) = X\}$

Miller–Steprāns paper

Two cardinals

- cov $\mathcal{M} = \min\{|F| : F \subseteq \omega^\omega \ \forall g \in \omega^\omega \ \exists f \in F \ \forall n \in \omega \ f(n) \neq g(n)\}$
- $\mathfrak{eq} = \min\{|F| : F \subseteq \omega^\omega$ bounded, $\forall g \in \omega^\omega \ \exists f \in F \ \forall n \in \omega \ f(n) \neq g(n)\}$

Theorem (Miller–Steprāns 2006)

- *If $\mathbb{G}$ is $\mathbb{R}^n$ or a countable product of finite groups then* cov$_{\mathbb{G}} = \mathfrak{eq}$
- *In particular, if $\mathbb{G} = 2^\omega$, then* cov$_{\mathbb{G}} = \mathfrak{eq}$
- *If $\mathbb{G} = \mathbb{Z}^\omega$, then* cov$_{\mathbb{G}} =$ cov $\mathcal{M}$

Miller–Steprāns paper

Two cardinals

- cov $\mathcal{M} = \min\{|F| : F \subseteq \omega^\omega \ \forall g \in \omega^\omega \ \exists f \in F \ \forall n \in \omega \ f(n) \neq g(n)\}$
- $\mathfrak{eq} = \min\{|F| : F \subseteq \omega^\omega$ bounded, $\forall g \in \omega^\omega \ \exists f \in F \ \forall n \in \omega \ f(n) \neq g(n)\}$

Theorem (Miller–Steprāns 2006)

- *If $\mathbb{G}$ is $\mathbb{R}^n$ or a countable product of finite groups then* cov$_{\mathbb{G}} = \mathfrak{eq}$
- *In particular, if $\mathbb{G} = 2^\omega$, then* cov$_{\mathbb{G}} = \mathfrak{eq}$
- *If $\mathbb{G} = \mathbb{Z}^\omega$, then* cov$_{\mathbb{G}} =$ cov $\mathcal{M}$

Miller–Steprāns paper

Two cardinals

- cov $\mathcal{M} = \min\{|F| : F \subseteq \omega^\omega\ \forall g \in \omega^\omega\ \exists f \in F\ \forall n \in \omega\ f(n) \neq g(n)\}$
- $\mathfrak{eq} = \min\{|F| : F \subseteq \omega^\omega$ bounded, $\forall g \in \omega^\omega\ \exists f \in F\ \forall n \in \omega\ f(n) \neq g(n)\}$

Theorem (Miller–Steprāns 2006)

- *If $\mathbb{G}$ is $\mathbb{R}^n$ or a countable product of finite groups then* $\mathrm{cov}_{\mathbb{G}} = \mathfrak{eq}$
- *In particular, if $\mathbb{G} = 2^\omega$, then* $\mathrm{cov}_{\mathbb{G}} = \mathfrak{eq}$
- *If $\mathbb{G} = \mathbb{Z}^\omega$, then* $\mathrm{cov}_{\mathbb{G}} = \mathrm{cov}\,\mathcal{M}$

Questions [Miller–Steprāns 2006]

1. Is it consistent to have a compact group $\mathbb{G}$ such that $\operatorname{cov}_{\mathbb{G}} > \mathfrak{eq}$?
2. Is it true that for any infinite compact group G we have $\operatorname{cov}_{\mathbb{G}} \geqslant \mathfrak{eq}$?
3. Is it true that for every non-discrete Polish group G we have $\operatorname{cov}_{\mathbb{G}} = \mathfrak{eq}$ or $\operatorname{cov}_{\mathbb{G}} = \operatorname{cov} \mathcal{M}$?
4. Let α_n be the natural action if the isometry group on $\mathbb{R}^n$. Is it true that $\operatorname{cov}_{\alpha_m} = \operatorname{cov}_{\alpha_n}$ for all m, n?

Definition

$\mathbb{G}$ is CLI if it admits a complete left-invariant metric.

Theorem (Dobrowolski–Marciszewski 2008)

If $\mathbb{G}$ is a Polish, not locally compact, CLI *group (in particular, if $\mathbb{G}$ is abelian), then $\operatorname{cov}_{\mathbb{G}} = \operatorname{cov} \mathcal{M}$.*

Miller–Steprāns questions

Questions [Miller–Steprāns 2006]

1. Is it consistent to have a compact group $\mathbb{G}$ such that $\operatorname{cov}_{\mathbb{G}} > \mathfrak{eq}$?
2. Is it true that for any infinite compact group G we have $\operatorname{cov}_{\mathbb{G}} \geqslant \mathfrak{eq}$?
3. Is it true that for every non-discrete Polish group G we have $\operatorname{cov}_{\mathbb{G}} = \mathfrak{eq}$ or $\operatorname{cov}_{\mathbb{G}} = \operatorname{cov} \mathcal{M}$?
4. Let α_n be the natural action if the isometry group on $\mathbb{R}^n$. Is it true that $\operatorname{cov}_{\alpha_m} = \operatorname{cov}_{\alpha_n}$ for all m, n?

Definition

$\mathbb{G}$ is CLI if it admits a complete left-invariant metric.

Theorem (Dobrowolski–Marciszewski 2008)

If $\mathbb{G}$ is a Polish, not locally compact, CLI *group (in particular, if $\mathbb{G}$ is abelian), then $\operatorname{cov}_{\mathbb{G}} = \operatorname{cov} \mathcal{M}$.*

Miller–Steprāns questions

Questions [Miller–Steprāns 2006]

1. Is it consistent to have a compact group $\mathbb{G}$ such that $\mathrm{cov}_{\mathbb{G}} > \mathfrak{eq}$?
2. Is it true that for any infinite compact group G we have $\mathrm{cov}_{\mathbb{G}} \geqslant \mathfrak{eq}$?
3. Is it true that for every non-discrete Polish group G we have $\mathrm{cov}_{\mathbb{G}} = \mathfrak{eq}$ or $\mathrm{cov}_{\mathbb{G}} = \mathrm{cov}\,\mathcal{M}$?
4. Let α_n be the natural action if the isometry group on $\mathbb{R}^n$. Is it true that $\mathrm{cov}_{\alpha_m} = \mathrm{cov}_{\alpha_n}$ for all m, n?

Definition

$\mathbb{G}$ is CLI if it admits a complete left-invariant metric.

Theorem (Dobrowolski–Marciszewski 2008)

If $\mathbb{G}$ is a Polish, not locally compact, CLI *group (in particular, if $\mathbb{G}$ is abelian), then* $\mathrm{cov}_{\mathbb{G}} = \mathrm{cov}\,\mathcal{M}$.

Strong measure zero

Definition

$\mathbf{Pr}(\mathbb{G}) = \{A \subseteq \mathbb{G} : \forall M \in \mathcal{M}\ A + M \neq \mathbb{G}\}$.

Uniformity

$\mathsf{cov}_{\mathbb{G}} = \mathsf{non}\,\mathbf{Pr}(\mathbb{G})$

Definition (Borel 1919)

A set A in a separable metric space has **strong measure zero** if for any sequence $\langle r_n : n \in \omega\rangle$ of radii there is a sequence $\langle x_n : n \in \omega\rangle$ such that $\{B(x_n, r_n) : n \in \omega\}$ covers A.

Theorem (Prikry 1973)

Let $\mathbb{G}$ be a separable group equipped with a left-invariant metric d. Then $\mathbf{Pr}(\mathbb{G}) \subseteq \mathbf{Smz}(\mathbb{G})$.

Strong measure zero

Definition

$\mathbf{Pr}(\mathbb{G}) = \{A \subseteq \mathbb{G} : \forall M \in \mathcal{M} \ A + M \neq \mathbb{G}\}$.

Uniformity

$\mathrm{cov}_{\mathbb{G}} = \mathrm{non}\ \mathbf{Pr}(\mathbb{G})$

Definition (Borel 1919)

A set A in a separable metric space has **strong measure zero** if for any sequence $\langle r_n : n \in \omega \rangle$ of radii there is a sequence $\langle x_n : n \in \omega \rangle$ such that $\{B(x_n, r_n) : n \in \omega\}$ covers A.

Theorem (Prikry 1973)

Let $\mathbb{G}$ be a separable group equipped with a left-invariant metric d. Then $\mathbf{Pr}(\mathbb{G}) \subseteq \mathbf{Smz}(\mathbb{G})$.

Strong measure zero

Definition

$\mathbf{Pr}(\mathbb{G}) = \{A \subseteq \mathbb{G} : \forall M \in \mathcal{M} \;\; A + M \neq \mathbb{G}\}$.

Uniformity

$\mathsf{cov}_{\mathbb{G}} = \mathsf{non}\, \mathbf{Pr}(\mathbb{G})$

Definition (Borel 1919)

A set A in a separable metric space has **strong measure zero** if for any sequence $\langle r_n : n \in \omega \rangle$ of radii there is a sequence $\langle x_n : n \in \omega \rangle$ such that $\{B(x_n, r_n) : n \in \omega\}$ covers A.

Theorem (Prikry 1973)

Let $\mathbb{G}$ be a separable group equipped with a left-invariant metric d. Then $\mathbf{Pr}(\mathbb{G}) \subseteq \mathbf{Smz}(\mathbb{G})$.

Strong measure zero vs. Prikry sets

Theorem (Prikry 1973)

Let $\mathbb{G}$ be a separable group equipped with a left-invariant metric d. Then $\mathbf{Pr}(\mathbb{G}) \subseteq \mathbf{Smz}(\mathbb{G})$.

Corollary

If $\mathbb{G}$ is a Polish group, then $\mathrm{cov}_{\mathbb{G}} \leqslant \mathrm{non}\,\mathbf{Smz}(\mathbb{G})$.

Corollary

If $\mathbb{G}$ is a Polish group, then $\mathrm{cov}(\mathcal{M}) \leqslant \mathrm{cov}_{\mathbb{G}} \leqslant \mathfrak{eq}$.

Proof.

- If $A \notin \mathbf{Pr}(\mathbb{G})$, then $\mathbb{G}$ is covered by $|A|$-many translates of a meager set.
- $\mathbb{G}$ contains (by the Perfect Set Theorem) a (uniform) copy of the Cantor space and therefore $\mathrm{non}(\mathbf{Smz}(X)) \leqslant \mathrm{non}(\mathbf{Smz}(2^\omega))$.
- $\mathrm{non}(\mathbf{Smz}(2^\omega)) = \mathfrak{eq}$ (Bartoszynski 1995) □

Strong measure zero vs. Prikry sets

Theorem (Prikry 1973)

Let $\mathbb{G}$ be a separable group equipped with a left-invariant metric d. Then $\mathbf{Pr}(\mathbb{G}) \subseteq \mathbf{Smz}(\mathbb{G})$.

Corollary

If $\mathbb{G}$ is a Polish group, then $\mathrm{cov}_{\mathbb{G}} \leqslant \mathrm{non}\,\mathbf{Smz}(\mathbb{G})$.

Corollary

If $\mathbb{G}$ is a Polish group, then $\mathrm{cov}(\mathcal{M}) \leqslant \mathrm{cov}_{\mathbb{G}} \leqslant \mathfrak{eq}$.

Proof.

- If $A \notin \mathbf{Pr}(\mathbb{G})$, then $\mathbb{G}$ is covered by $|A|$-many translates of a meager set.
- $\mathbb{G}$ contains (by the Perfect Set Theorem) a (uniform) copy of the Cantor space and therefore $\mathrm{non}(\mathbf{Smz}(X)) \leqslant \mathrm{non}(\mathbf{Smz}(2^\omega))$.
- $\mathrm{non}(\mathbf{Smz}(2^\omega)) = \mathfrak{eq}$ (Bartoszynski 1995) $\square$

Strong measure zero vs. Prikry sets

Theorem (Prikry 1973)

Let $\mathbb{G}$ *be a separable group equipped with a left-invariant metric* d*. Then* $\mathbf{Pr}(\mathbb{G}) \subseteq \mathbf{Smz}(\mathbb{G})$*.*

Corollary

If $\mathbb{G}$ *is a Polish group, then* $\mathrm{cov}_{\mathbb{G}} \leqslant \mathrm{non}\,\mathbf{Smz}(\mathbb{G})$*.*

Corollary

If $\mathbb{G}$ *is a Polish group, then* $\mathrm{cov}(\mathcal{M}) \leqslant \mathrm{cov}_{\mathbb{G}} \leqslant \mathfrak{eq}$*.*

Proof.

- If $A \notin \mathbf{Pr}(\mathbb{G})$, then $\mathbb{G}$ is covered by $|A|$-many translates of a meager set.
- $\mathbb{G}$ contains (by the Perfect Set Theorem) a (uniform) copy of the Cantor space and therefore $\mathrm{non}(\mathbf{Smz}(X)) \leqslant \mathrm{non}(\mathbf{Smz}(2^\omega))$.
- $\mathrm{non}(\mathbf{Smz}(2^\omega)) = \mathfrak{eq}$ (Bartoszynski 1995) □

Galvin–Mycielski–Solovay Theorem

Theorem (Galvin–Mycielski–Solovay 1973)

$\mathbf{Pr}(\mathbb{R}) = \mathbf{Smz}(\mathbb{R})$

Question

Which Polish groups satisfy Galvin–Mycielski–Solovay Theorem?

Theorem (Kysiak 2000, Fremlin 2008, Zindulka 2010)

$\mathbf{Pr}(\mathbb{G}) = \mathbf{Smz}(\mathbb{G})$ *for every locally compact group* $\mathbb{G}$.

Galvin–Mycielski–Solovay Theorem

Theorem (Galvin–Mycielski–Solovay 1973)

$\mathbf{Pr}(\mathbb{R}) = \mathbf{Smz}(\mathbb{R})$

Question

Which Polish groups satisfy Galvin–Mycielski–Solovay Theorem?

Theorem (Kysiak 2000, Fremlin 2008, Zindulka 2010)

$\mathbf{Pr}(\mathbb{G}) = \mathbf{Smz}(\mathbb{G})$ *for every locally compact group* $\mathbb{G}$.

Galvin–Mycielski–Solovay Theorem

Theorem (Galvin–Mycielski–Solovay 1973)

$\mathbf{Pr}(\mathbb{R}) = \mathbf{Smz}(\mathbb{R})$

Question

Which Polish groups satisfy Galvin–Mycielski–Solovay Theorem?

Theorem (Kysiak 2000, Fremlin 2008, Zindulka 2010)

$\mathbf{Pr}(\mathbb{G}) = \mathbf{Smz}(\mathbb{G})$ *for every locally compact group* $\mathbb{G}$*.*

Miller–Steprāns questions revisited

Questions

1. Is it consistent to have a compact group $\mathbb{G}$ such that $\operatorname{cov}_{\mathbb{G}} > \mathfrak{eq}$?
2. Is it true that for any infinite compact group $\mathbb{G}$ we have $\operatorname{cov}_{\mathbb{G}} \geqslant \mathfrak{eq}$?

Theorem

$\operatorname{cov}_{\mathbb{G}} = \operatorname{non} \mathbf{Pr}(\mathbb{G}) = \mathfrak{eq}$ *for every locally compact group* $\mathbb{G}$.

Proof.

Only need $\operatorname{non} \mathbf{Smz}(\mathbb{G}) \geqslant \mathfrak{eq}$.
Suppose X be compact. There is a continuous mapping $f : 2^\omega \to X$ onto X.
It is of course uniformly continuous, so
$\operatorname{non}(\mathbf{Smz}(X)) \geqslant \operatorname{non}(\mathbf{Smz}(2^\omega)) = \mathfrak{eq}$. □

Question

Which Polish groups satisfy Galvin–Mycielski–Solovay Theorem?

Miller–Steprāns questions revisited

Questions

1. Is it consistent to have a compact group $\mathbb{G}$ such that $\operatorname{cov}_{\mathbb{G}} > \mathfrak{eq}$?
2. Is it true that for any infinite compact group $\mathbb{G}$ we have $\operatorname{cov}_{\mathbb{G}} \geqslant \mathfrak{eq}$?

Theorem

$\operatorname{cov}_{\mathbb{G}} = \operatorname{non}\mathbf{Pr}(\mathbb{G}) = \mathfrak{eq}$ *for every locally compact group* $\mathbb{G}$.

Proof.

Only need $\operatorname{non}\mathbf{Smz}(\mathbb{G}) \geqslant \mathfrak{eq}$.
Suppose X be compact. There is a continuous mapping $f : 2^\omega \to X$ onto X.
It is of course uniformly continuous, so
$\operatorname{non}(\mathbf{Smz}(X)) \geqslant \operatorname{non}(\mathbf{Smz}(2^\omega)) = \mathfrak{eq}$. $\square$

Question

Which Polish groups satisfy Galvin–Mycielski–Solovay Theorem?

Miller–Steprāns questions revisited

Questions

1. Is it consistent to have a compact group $\mathbb{G}$ such that $\mathrm{cov}_{\mathbb{G}} > \mathfrak{eq}$?
2. Is it true that for any infinite compact group $\mathbb{G}$ we have $\mathrm{cov}_{\mathbb{G}} \geqslant \mathfrak{eq}$?

Theorem

$\mathrm{cov}_{\mathbb{G}} = \mathrm{non}\,\mathbf{Pr}(\mathbb{G}) = \mathfrak{eq}$ *for every locally compact group* $\mathbb{G}$.

Proof.

Only need $\mathrm{non}\,\mathbf{Smz}(\mathbb{G}) \geqslant \mathfrak{eq}$.
Suppose X be compact. There is a continuous mapping $f : 2^\omega \to X$ onto X.
It is of course uniformly continuous, so
$\mathrm{non}(\mathbf{Smz}(X)) \geqslant \mathrm{non}(\mathbf{Smz}(2^\omega)) = \mathfrak{eq}$. $\square$

Question

Which Polish groups satisfy Galvin–Mycielski–Solovay Theorem?

Miller–Steprāns questions revisited

Questions

1. Is it consistent to have a compact group $\mathbb{G}$ such that $\mathrm{cov}_{\mathbb{G}} > \mathfrak{eq}$?
2. Is it true that for any infinite compact group $\mathbb{G}$ we have $\mathrm{cov}_{\mathbb{G}} \geqslant \mathfrak{eq}$?

Theorem

$\mathrm{cov}_{\mathbb{G}} = \mathrm{non}\,\mathbf{Pr}(\mathbb{G}) = \mathfrak{eq}$ *for every locally compact group* $\mathbb{G}$.

Proof.

Only need $\mathrm{non}\,\mathbf{Smz}(\mathbb{G}) \geqslant \mathfrak{eq}$.
Suppose X be compact. There is a continuous mapping $f : 2^\omega \to X$ onto X.
It is of course uniformly continuous, so
$\mathrm{non}(\mathbf{Smz}(X)) \geqslant \mathrm{non}(\mathbf{Smz}(2^\omega)) = \mathfrak{eq}$. $\square$

Question

Which Polish groups satisfy Galvin–Mycielski–Solovay Theorem?

Hard work

GMS Theorem for non-locally compact groups?

Definition

$\mathbb{G}$ is GMS if $\mathbf{Pr}(\mathbb{G}) = \mathbf{Smz}(\mathbb{G})$ in ZFC.

Definition

$\mathbb{G}$ is **weakly** GMS if for every closed nowhere dense $M \subseteq \mathbb{G}$ there is a $\langle \varepsilon_n : n \in \omega \rangle$ $\forall \langle U_n : n \in \omega \rangle$ such that $\operatorname{diam} U_n < \varepsilon_n$ there is a $g \in \mathbb{G}$ such that $(g \cdot \bigcup_{n \in \omega} U_n) \cap M$ is not dense in M.

Theorem

Every Polish GMS group is weakly GMS.

Theorem ($\operatorname{cov}(\mathcal{M}) = \mathfrak{c}$)

If $\mathbb{G}$ is Polish and not weakly GMS, then $\mathbf{Pr}(\mathbb{G}) \neq \mathbf{Smz}(\mathbb{G})$.

GMS Theorem for non-locally compact groups?

Definition

$\mathbb{G}$ is GMS if $\mathbf{Pr}(\mathbb{G}) = \mathbf{Smz}(\mathbb{G})$ in ZFC.

Definition

$\mathbb{G}$ is **weakly** GMS if for every closed nowhere dense $M \subseteq \mathbb{G}$ there is a $\langle \varepsilon_n : n \in \omega \rangle$ $\forall \langle U_n : n \in \omega \rangle$ such that $\operatorname{diam} U_n < \varepsilon_n$ there is a $g \in \mathbb{G}$ such that $(g \cdot \bigcup_{n \in \omega} U_n) \cap M$ is not dense in M.

Theorem

Every Polish GMS *group is weakly* GMS.

Theorem ($\operatorname{cov}(\mathcal{M}) = \mathfrak{c}$)

If $\mathbb{G}$ is Polish and not weakly GMS*, then* $\mathbf{Pr}(\mathbb{G}) \neq \mathbf{Smz}(\mathbb{G})$.

GMS Theorem for non-locally compact groups?

Definition

$\mathbb{G}$ is GMS if **Pr**$(\mathbb{G}) =$ **Smz**$(\mathbb{G})$ in ZFC.

Definition

$\mathbb{G}$ is **weakly** GMS if for every closed nowhere dense $M \subseteq \mathbb{G}$ there is a $\langle \varepsilon_n : n \in \omega \rangle$ $\forall \langle U_n : n \in \omega \rangle$ such that $\operatorname{diam} U_n < \varepsilon_n$ there is a $g \in \mathbb{G}$ such that $(g \cdot \bigcup_{n \in \omega} U_n) \cap M$ is not dense in M.

Theorem

Every Polish GMS *group is weakly* GMS*.*

Theorem ($\operatorname{cov}(\mathcal{M})=\mathfrak{c}$)

If $\mathbb{G}$ is Polish and not weakly GMS*, then* **Pr**$(\mathbb{G}) \neq$ **Smz**$(\mathbb{G})$*.*

GMS Theorem for non-locally compact groups may fail

Theorem

The group $\mathbb{Z}^\omega$ is not weakly GMS.

Theorem

Consistently, $\mathbf{Pr}(\mathbb{Z}^\omega) \neq \mathbf{Smz}(\mathbb{Z}^\omega)$.

Remark (Borel Conjecture)

Consistently, $\mathbf{Pr}(\mathbb{G}) = \mathbf{Smz}(\mathbb{G})$ for all Polish groups.

GMS Theorem for non-locally compact groups may fail

Theorem

The group $\mathbb{Z}^\omega$ *is not weakly* GMS.

Theorem

Consistently, $\mathbf{Pr}(\mathbb{Z}^\omega) \neq \mathbf{Smz}(\mathbb{Z}^\omega)$.

Remark (Borel Conjecture)

Consistently, $\mathbf{Pr}(\mathbb{G}) = \mathbf{Smz}(\mathbb{G})$ for all Polish groups.

GMS Theorem for non-locally compact groups may fail

Theorem

The group $\mathbb{Z}^\omega$ *is not weakly* GMS.

Theorem

Consistently, $\mathbf{Pr}(\mathbb{Z}^\omega) \neq \mathbf{Smz}(\mathbb{Z}^\omega)$.

Remark (Borel Conjecture)

Consistently, $\mathbf{Pr}(\mathbb{G}) = \mathbf{Smz}(\mathbb{G})$ for all Polish groups.

Third question revisited

Question

Is it true that for every non-discrete Polish group G we have $\mathrm{cov}_{\mathbb{G}} = \mathfrak{eq}$ or $\mathrm{cov}_{\mathbb{G}} = \mathrm{cov}\,\mathcal{M}$?

Question rephrased

Is it true that for every non-locally compact Polish group G we have $\mathrm{non}\,\mathbf{Pr}(\mathbb{G}) = \mathrm{cov}\,\mathcal{M}$?

Theorem

A CLI *Polish group is either locally compact, or else contains a uniform copy of ω^ω.*

Corollary

$\mathrm{non}\,\mathbf{Pr}(\mathbb{G}) = \mathrm{non}\,\mathbf{Smz}(\mathbb{G}) = \mathrm{cov}\,\mathcal{M}$ *for every non-locally compact* CLI *Polish group $\mathbb{G}$.*

Third question revisited

Question

Is it true that for every non-discrete Polish group G we have $\mathrm{cov}_{\mathbb{G}} = \mathfrak{eq}$ or $\mathrm{cov}_{\mathbb{G}} = \mathrm{cov}\,\mathcal{M}$?

Question rephrased

Is it true that for every non-locally compact Polish group G we have $\mathrm{non}\,\mathbf{Pr}(\mathbb{G}) = \mathrm{cov}\,\mathcal{M}$?

Theorem

A CLI *Polish group is either locally compact, or else contains a uniform copy of* ω^ω*.*

Corollary

$\mathrm{non}\,\mathbf{Pr}(\mathbb{G}) = \mathrm{non}\,\mathbf{Smz}(\mathbb{G}) = \mathrm{cov}\,\mathcal{M}$ *for every non-locally compact* CLI *Polish group* $\mathbb{G}$*.*

Third question revisited

Question

Is it true that for every non-discrete Polish group G we have $\mathrm{cov}_{\mathbb{G}} = \mathfrak{eq}$ or $\mathrm{cov}_{\mathbb{G}} = \mathrm{cov}\,\mathcal{M}$?

Question rephrased

Is it true that for every non-locally compact Polish group G we have $\mathrm{non}\,\mathbf{Pr}(\mathbb{G}) = \mathrm{cov}\,\mathcal{M}$?

Theorem

A CLI *Polish group is either locally compact, or else contains a uniform copy of* ω^ω*.*

Corollary

$\mathrm{non}\,\mathbf{Pr}(\mathbb{G}) = \mathrm{non}\,\mathbf{Smz}(\mathbb{G}) = \mathrm{cov}\,\mathcal{M}$ *for every non-locally compact* CLI *Polish group* $\mathbb{G}$*.*

Last question revisited

Definition

Let α be an action of a Polish group $\mathbb{G}$ on a Polish space X. Let

$$\mathbf{Pr}(\alpha) = \{A \subseteq \mathbb{G} : \forall M \in \mathcal{M}\ \alpha(A \times M) \neq X\}$$

Question

Let α_n be the natural action of the isometry group on $\mathbb{R}^n$. Is it true that $\operatorname{non}\mathbf{Pr}(\alpha_m) = \operatorname{non}\mathbf{Pr}(\alpha_n)$ for all m, n?

Theorem

If α is an action of a σ-compact Polish group $\mathbb{G}$ on a Polish space, then $\mathbf{Smz}(\mathbb{G}) \subseteq \mathbf{Pr}(\alpha)$. Consequently $\operatorname{non}\mathbf{Pr}(\alpha) \geqslant \mathfrak{eq}$.

Corollary (Answer to the question)

Yes, and they are all equal to $\mathfrak{eq}$.

Last question revisited

Definition

Let α be an action of a Polish group $\mathbb{G}$ on a Polish space X. Let

$$\mathbf{Pr}(\alpha) = \{A \subseteq \mathbb{G} : \forall M \in \mathcal{M}\ \alpha(A \times M) \neq X\}$$

Question

Let α_n be the natural action of the isometry group on $\mathbb{R}^n$. Is it true that non $\mathbf{Pr}(\alpha_m) =$ non $\mathbf{Pr}(\alpha_n)$ for all m, n?

Theorem

If α is an action of a σ-compact Polish group $\mathbb{G}$ on a Polish space, then $\mathbf{Smz}(\mathbb{G}) \subseteq \mathbf{Pr}(\alpha)$. *Consequently* non $\mathbf{Pr}(\alpha) \geqslant \mathfrak{eq}$.

Corollary (Answer to the question)

Yes, and they are all equal to $\mathfrak{eq}$.

Last question revisited

Definition

Let α be an action of a Polish group $\mathbb{G}$ on a Polish space X. Let

$$\mathbf{Pr}(\alpha) = \{A \subseteq \mathbb{G} : \forall M \in \mathcal{M}\ \alpha(A \times M) \neq X\}$$

Question

Let α_n be the natural action of the isometry group on $\mathbb{R}^n$. Is it true that non $\mathbf{Pr}(\alpha_m) =$ non $\mathbf{Pr}(\alpha_n)$ for all m, n?

Theorem

If α is an action of a σ-compact Polish group $\mathbb{G}$ on a Polish space, then $\mathbf{Smz}(\mathbb{G}) \subseteq \mathbf{Pr}(\alpha)$*. Consequently* non $\mathbf{Pr}(\alpha) \geqslant \mathfrak{eq}$.

Corollary (Answer to the question)

Yes, and they are all equal to $\mathfrak{eq}$.

Two mysteries

Theorem

The group $\mathbb{Z}^\omega$ *is not weakly* GMS.

Question

Is is true that no Polish non-locally compact group is weakly GMS?

Corollary

$\operatorname{non}\mathbf{Pr}(\mathbb{G}) = \operatorname{non}\mathbf{Smz}(\mathbb{G}) = \operatorname{cov}\mathcal{M}$ *for every non-locally compact* CLI *Polish group* $\mathbb{G}$.

Question

Can one drop the CLI assumption?

Two mysteries

Theorem

The group $\mathbb{Z}^\omega$ *is not weakly* GMS*.*

Question

Is is true that no Polish non-locally compact group is weakly GMS?

Corollary

non $\mathbf{Pr}(\mathbb{G}) =$ non $\mathbf{Smz}(\mathbb{G}) =$ cov $\mathcal{M}$ *for every non-locally compact* CLI *Polish group* $\mathbb{G}$*.*

Question

Can one drop the CLI assumption?

Two mysteries

Theorem

The group $\mathbb{Z}^\omega$ *is not weakly* GMS*.*

Question

Is is true that no Polish non-locally compact group is weakly GMS?

Corollary

$\operatorname{non}\mathbf{Pr}(\mathbb{G}) = \operatorname{non}\mathbf{Smz}(\mathbb{G}) = \operatorname{cov}\mathcal{M}$ *for every non-locally compact* CLI *Polish group* $\mathbb{G}$*.*

Question

Can one drop the CLI assumption?

Two mysteries

Theorem

The group $\mathbb{Z}^\omega$ *is not weakly* GMS*.*

Question

Is is true that no Polish non-locally compact group is weakly GMS?

Corollary

$\operatorname{non}\mathbf{Pr}(\mathbb{G}) = \operatorname{non}\mathbf{Smz}(\mathbb{G}) = \operatorname{cov}\mathcal{M}$ *for every non-locally compact* CLI *Polish group* $\mathbb{G}$*.*

Question

Can one drop the CLI assumption?