Axial functions - some results

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Def. Function $f : X \times Y \to X \times Y$ is axial if
$f(x, y) = (x, g(x, y))$ for some $g\colon X \times Y \to Y$ f is vertical
or
$f(x, y) = (g(x, y), y)$ for some $g\colon X \times Y \to X$. f is horizontal

Def. Function $f : X_1 \times ... \times X_n \to X_1 \times ... \times X_n$ is axial if there exists $i \in \{1, ..., n\}$ such that $f(x_1, ..., x_n) = (x_1, .., x_{i-1}, g(x_1, ..., x_n), x_{i+1}, .., x_n)$ for some $g : X_1 \times ... \times X_n \to X_i$.

General question: which functions are compositions of axial functions

First questions about axial functions by Banach and Ulam in Scottisch Book

Finite sets

Thm. Every function $f : X \times Y \to X \times Y$, where X, Y are finite, is a superposition of five axial functions $f = h_5 \circ ... \circ h_1$. Additionally we may demand that h_1 is ,say, horizontal.

Three axial functions are not enough

Ex. There is a fnction which is not a composition of four axial functions provided that the first axial function is horizontal.

Question: Are four axial functions enough to obtain any function for X, Y finite ?

Infinite sets

Problem (Banach 1935) Is every permutation of $\mathbb{N}^2$ a composition of finite number of axial permutation ?

Thm. Every function $f : X \times Y \to X \times Y$, where X or Y is infinite, is a composition of three axial functions. easy

Thm.Every function $f : X_1 \times ... \times X_n \to X_1 \times ... \times X_n$, where X_i are infinite, is a composition of $n+1$ axial functions.

Thm. Every function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a superposition of three measurable axial functions. Every Borel function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a superposition of three Borel axial functions.

Continuity

Thm. (Eggleston 55) Every homeomorphism of $\mathbb{R}^2$ is a pointwise limit of a composition of axial homeomorphisms (but not uniform limit).

Thm. (Eggleston 55) Every homeomorphism of a unit square $[0,1]^2$ that is identity on the boundary of $[0,1]^2$ can be approximated in supremum metric by a compositions of finite number of axial homeomorphisms.

But the number of these axial homeomorphisms can not be bounded.

Thm. (Szyszkowski 2010) There is a continous (not 1-1) function $f : [0,1]^2 \to [0,1]^2$ such that

$$|f - \varphi_n \circ \varphi_{n-1} \circ ... \varphi_1| \geq \frac{1}{10}$$

for every axial continuous φ_i and every n.

almost Thm. Homeomorphisms of $[0,1]^3$ that leave frontier points fixed can be approximated in supremum metric by a compositions of finite number of axial homeomorphisms.

some other results/questions

A linear function on $\mathbb{R}^n$ is a composition of axial linear functions.not difficult

A piecewise linear function on $\mathbb{R}^n$ is a composition of axial piecewiese linear functions - still need to be verified.

Polynomial function on $\mathbb{R}^2$ is not a composition of axial continuous functions.