Hyperstability of some functional equation

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We deal with the general linear functional equation

$$\sum_{i=1}^{m} A_i f(\sum_{j=1}^{n} a_{ij} x_j) + A = 0, \tag{1}$$

where $A, a_{ij} \in \mathbb{F}$, $A_i \in \mathbb{F} \setminus \{0\}$, $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$.

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Fréchet equation	$\Delta^n_{x_1 \cdots x_n} f(x) = 0,$

where $a, b, A, B, p \in \mathbb{F} \setminus \{0, 1\}$ and $n \in \mathbb{N}$.

A^B the family of all functions mapping a set $B \neq \emptyset$ into a set $A \neq \emptyset$

(H1) X is a nonempty set, Y is a Banach space, $f_1, \ldots, f_j \colon X \to X$ and $L_1, \ldots, L_j \colon X \to \mathbb{R}_+ := [0, \infty)$ are given, and $\mathcal{T} \colon Y^X \to Y^X$ is an operator satisfying the inequality

$$\left\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\right\| \leq \sum_{i=1}^{j} L_i(x)\left\|\xi(f_i(x)) - \mu(f_i(x))\right\|, \qquad \xi, \mu \in Y^X, x \in X.$$

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(H2) $\Lambda \colon {\mathbb{R}_+}^X \to {\mathbb{R}_+}^X$ is defined by

$$\Lambda\delta(x) := \sum_{i=1}^{j} L_i(x)\delta(f_i(x)), \qquad \delta \in {\mathbb{R}_+}^X, x \in X.$$

J. Brzdęk, J. Chudziak, Z. Páles, *A fixed point approach to stability of functional equations, Nonlinear Anal.* **74***(2011), 6728-6732.*

Theorem 1

Let hypotheses (H1), (H2) *be valid and the functions* $\varepsilon\colon X \to \mathbb{R}_+$ *and* $\varphi\colon X \to Y$ *fulfil the following two conditions*

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$$\left\|\mathcal{T}\varphi(x) - \varphi(x)\right\| \le \varepsilon(x), \qquad x \in X,$$

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) < \infty, \qquad x \in X.$$

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Moreover

$$\psi(x) := \lim_{n\to\infty} \mathcal{T}^n\varphi(x), \qquad x \in X.$$

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$$\sum_{i=1}^{m} A_i f(\sum_{j=1}^{n} a_{ij} x_j) + A = 0, \tag{1}$$

where $f : X \to Y$ and $A, a_{ij} \in \mathbb{F}$, $A_i \in \mathbb{F} \setminus \{0\}$, $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$.

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From now on, we assume that X, Y are the normed spaces over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and the coefficients in the equation (1) are such that

$$A = 0 \text{ or } \Big(A \neq 0 \text{ and } \sum_{i=1}^{m} A_i \neq 0\Big).$$

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Denote

$$\begin{aligned} X_0 &:= X \setminus \{0\}, \\ \mathbb{F}_0 &:= \mathbb{F} \setminus \{0\}, \\ \mathbb{N}_{\leq k} &:= \{1, \dots, k\}, \\ \mathbb{N}_k &:= \{l \in \mathbb{N} \cup \{0\} : l \geq k\}, \qquad k \in \mathbb{N} \cup \{0\}. \end{aligned}$$

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Let the functions $g : X \to Y$, $\omega : \mathbb{F}_0 \to \mathbb{R}_+$, $\theta \colon X_0^n \to \mathbb{R}_+$ *satisfy*

$$\theta(\beta x_1, \ldots, \beta x_n) \leq \omega(\beta)\theta(x_1, \ldots, x_n) \qquad \beta \in \mathbb{F}_0,\ x_1, \ldots, x_n \in X_0, \tag{2}$$

and

$$\Big\| \sum_{i=1}^{m} A_i g(\sum_{j=1}^{n} a_{ij} x_j) + A \Big\| \leq \theta(x_1, \ldots, x_n), \qquad x_1, \ldots, x_n \in X_0. \tag{3}$$

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If there exist $\emptyset \neq I \subset \mathbb{N}_{\leq m}$ *and the sequence* $\{(c_1^k, \dots, c_n^k)\}_{k \in \mathbb{N}}$ *of elements of* $\mathbb{F}_0^n$ *such that*

$$\beta_i^k := \sum_{j=1}^{n} a_{ij} c_j^k \in F_0, \qquad i \in \mathbb{N}_{\leq m},\ k \in \mathbb{N},$$

$$\beta_i^k = 1,\ i \in I, \quad A_I := \sum_{i \in I} A_i \neq 0, \quad \lim_{k \to \infty} \sum_{i \notin I} \Big| \frac{A_i}{A_I} \Big| \omega(\beta_i^k) < 1,$$

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then g satisfies

$$\sum_{i=1}^{m} A_i g\Big(\sum_{j=1}^{n} a_{ij} x_j\Big) + A = 0, \qquad x_1, \dots, x_n \in X_0. \tag{4}$$

Proof. Without loss of generality we can assume that Y is a Banach space.
There exists $\alpha \in (0, 1)$ and $k_0 \in \mathbb{N}$ such that

$$\gamma_k := \sum_{i \notin I} \left| \frac{A_i}{A_I} \right| \omega(\beta_i^k) < \alpha, \quad k \geq k_0. \tag{5}$$

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First assume that $A = 0$. Taking $x \in X_0$, $k \geq k_0$ and substituting $x_j = c_j^k x$, $j \in \mathbb{N}_{\leq n}$ in (3) we have

$$\|g(x) - \sum_{i \notin I} \frac{-A_i}{A_I} g(\beta_i^k x)\| \leq \frac{\theta(c_1^k x, \ldots, c_n^k x)}{|A_I|}, \qquad x \in X_0. \tag{6}$$

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$$\begin{aligned} \mathcal{T}\xi(x) &:= \sum_{i \notin I} \frac{-A_i}{A_I} \xi(\beta_i^k x), \qquad \xi \in Y^{X_0}, x \in X_0, \\ \Lambda\delta(x) &:= \sum_{i \notin I} \left|\frac{-A_i}{A_I}\right| \delta(\beta_i^k x), \qquad \delta \in {\mathbb{R}_+}^{X_0}, x \in X_0, \\ \varepsilon(x) &:= \frac{\theta(c_1^k x, \ldots, c_n^k x)}{|A_I|}, \qquad x \in X_0. \end{aligned}$$

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Using (5), we get

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} (\Lambda^n \varepsilon)(x) \leq \varepsilon(x) \sum_{n=0}^{\infty} \gamma_k^n = \frac{\varepsilon(x)}{1 - \gamma_k}, \qquad x \in X_0.$$

There exists a unique fixed point $G_k : X_0 \to Y$ of $\mathcal{T}$ such that

$$\|g(x) - G_k(x)\| \leq \frac{\theta(c_1^k x, \ldots, c_n^k x)}{|A_l|(1 - \gamma_k)}, \qquad x \in X_0. \tag{7}$$

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It is easy to show that for every $l \in \mathbb{N}_0$ and every $x_1, \dots, x_n \in X_0$

$$\Big\| \sum_{i=1}^{m} A_i(\mathcal{T}^l g)(\sum_{j=1}^{n} a_{ij}x_j)\Big\| \leq \gamma_k^l \theta(x_1, \dots, x_n).$$

G_k satisfies the equation (1) (with $A = 0$).
We obtain the sequence $\{G_k\}_{k\in\mathbb{N}_{k_0}}$ of functions satisfying (1). Letting in (7) $k \to \infty$, we get that g satisfies (4).

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If $A \neq 0$ and $\sum_{i=1}^m A_i \neq 0$ we define a function $h : X_0 \to Y$ in the following way

$$h(x) := g(x) + \frac{A}{\sum_{i=1}^m A_i}.$$

From (3)

$$\Big\| \sum_{i=1}^{m} A_i h(\sum_{j=1}^{n} a_{ij} x_j) \Big\| \leq \theta(x_1, \dots, x_n), \qquad x_1, \dots, x_n \in X_0,$$

and consequently, the function h satisfies (1) with $A = 0$, and hence g is a solution of (4).

Applying Theorem 2 we can obtain the following result for the particular form of θ, namely for

$$\theta(x_1, \ldots, x_n) = C \sqcap_{j=1}^{n} \|x_j\|^{k_j}$$

with $C > 0$, $k_j \in \mathbb{R}$ such that $\sum_{j=1}^{n} k_j < 0$.

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Notice that defined this way the function θ fulfills the condition

$$\theta(\beta x_1, \ldots, \beta x_n) \leq \omega(\beta)\theta(x_1, \ldots, x_n), \qquad \beta \in \mathbb{F}_0,\ x_1, \ldots, x_n \in X_0. \tag{2}$$

with $\omega(\beta) = |\beta|^{\sum_{j=1}^{n} k_j}$ for $\beta \in \mathbb{F}_0$.

Theorem 3

Assume that $g : X \to Y$ fulfills

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If there exist $i_0 \in \mathbb{N}_{\leq m}$ *and* $j_0 \in \mathbb{N}_{\leq n}$ *such that*

$$\sum_{j \neq j_0} k_j < 0, \quad a_{i_0 j_0} \neq 0, \quad \sum_{j \neq j_0} a_{i_0 j} = 0 \quad \text{and} \quad \sum_{j \neq j_0} a_{ij} \neq 0 \text{ for } i \neq i_0,$$

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$$\sum_{j \ne j_0} k_j < 0, \quad a_{i_0 j_0} \ne 0, \quad \sum_{j \ne j_0} a_{i_0 j} = 0 \quad \text{and} \quad \sum_{j \ne j_0} a_{ij} \ne 0 \text{ for } i \ne i_0,$$

then g satisfies (4).

Proof. Define the sequence $\{(c_1^k, \ldots, c_n^k)\}_{k \in \mathbb{N}}$ as follows

$$c_j^k := \begin{cases} k & \text{for } j \ne j_0 \\ \frac{1}{a_{i_0 j_0}} & \text{for } j = j_0 \end{cases} .$$

Theorem 3

Assume that $g : X \to Y$ fulfills

$$\|\sum_{i=1}^{m} A_i g(\sum_{j=1}^{n} a_{ij}x_j) + A\| \leq C \sqcap_{j=1}^{n} \|x_j\|^{k_j}, \qquad x_1, \ldots, x_n \in X_0$$

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The assumptions of Theorem 2 are fulfilled with $I = \{i_0\}$, which completes the proof.

Theorem 3

Assume that $g : X \to Y$ fulfills

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then g satisfies (4).

From Theorem 3 we obtain that linear equation (especially Cauchy, Jensen equation), quadratic equation, p-Write equation are θ-hyperstable with $\theta(x, y) = C\|x\|^{k_1}\|y\|^{k_2}$ $C \geq 0$ and $k_1 + k_2 < 0$.

$$\Delta^n_{h_1 \cdots h_n} f(x) = 0, \tag{8}$$

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$$\Delta_h f(x) = \Delta^1_h f(x) := f(x+h) - f(x), \qquad x, h \in X,$$

$$\Delta^{n+1}_{h_1 \dots h_{n+1}} f(x) = \Delta_{h_{n+1}}\Big(\Delta^n_{h_1 \cdots h_n} f(x)\Big) \qquad x, h_1 \dots h_{n+1} \in X, \quad n \in \mathbb{N}.$$

Fréchet functional equation

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Corollary 4

Assume that $n \in \mathbb{N}_2$ and $g : X \to Y$ fulfills

$$\|\Delta^{n-1}_{x_2 \cdots x_n} g(x_1)\| \le \theta_l(x_1, \ldots, x_n), \qquad l \in \{1,2\}, \quad x_1, \ldots, x_n \in X_0$$

$$\text{where} \quad \begin{array}{ll} \theta_1(x_1, \ldots, x_n) := \left(\sum_{j=1}^n C_j \|x_j\|^{k_j}\right)^w & \text{with } C_j > 0,\ wk_j < 0 \text{ for } j \in \{1, \ldots, n\}, \\ \theta_2(x_1, \ldots, x_n) := C \sqcap_{j=1}^n \|x_j\|^{k_j} & \text{with } C > 0,\ \sum_{j=1}^n k_j < 0. \end{array}$$

Corollary 4

Assume that $n \in \mathbb{N}_2$ *and* $g : X \to Y$ *fulfills*

$$\|\Delta^{n-1}_{x_2 \cdots x_n} g(x_1)\| \leq \theta_l(x_1, \ldots, x_n), \qquad l \in \{1, 2\}, \quad x_1, \ldots, x_n \in X_0$$

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$$\begin{aligned} \theta_1(x_1, \ldots, x_n) &:= \left(\textstyle\sum_{j=1}^{n} C_j \|x_j\|^{k_j}\right)^w && \text{with } C_j > 0,\ wk_j < 0 \text{ for } j \in \{1, \ldots, n\}, \\ \theta_2(x_1, \ldots, x_n) &:= C \textstyle\sqcap_{j=1}^{n} \|x_j\|^{k_j} && \text{with } C > 0,\ \textstyle\sum_{j=1}^{n} k_j < 0. \end{aligned}$$

Then g *is a solution of the equation*

$$\Delta^{n-1}_{x_2 \cdots x_n} g(x_1) = 0, \qquad x_1, \ldots, x_n \in X_0.$$

Notice that putting $x = 0$ in

$$\Delta^n_{h_1 \cdots h_n} f(x) = 0, \tag{8}$$

and assuming that $f(0) = 0$ we obtain with $n = 2$ Cauchy equation

$$f(x + y) = f(x) + f(y),$$

and with $n = 3$ the equation

$$f(x + y + z) + f(x) + f(y) + f(z) = f(x + y) + f(x + z) + f(y + z). \tag{9}$$

Denote

$$F(f, x_1, \ldots, x_n) = \Delta^n_{x_1 \cdots x_n} f(0) + (-1)^{n+1} f(0), \qquad f \in Y^X, x_1, \ldots, x_n \in X.$$

$$F(g, x_1, \ldots, x_n) = 0 \tag{10}$$

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Corollary 5

Assume that $n\in\mathbb{N}_2$ *and* $g:X\to Y$ *fulfills*

$$\|F(g,x_1,\dots,x_n)\|\le\theta_l(x_1,\dots,x_n),\qquad l\in\{1,2\},\ \ x_1,\dots,x_n\in X_0$$

where $\theta_1(x_1,\dots,x_n):=(\sum_{j=1}^n C_j\|x_j\|^{k_j})^w$ *with* $C_j>0,\ wk_j<0$ *for* $j\in\{1,\dots,n\}$,
$\theta_2(x_1,\dots,x_n):=C\sqcap_{j=1}^n\|x_j\|^{k_j}$ *with* $C>0,\ \sum_{j=1}^n k_j<0$.

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Then g is a solution of the equation

$$F(g, x_1, \ldots, x_n) = 0, \qquad x_1, \ldots, x_n \in X_0.$$

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