Differentiation, representation, localization

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Notation

Let I denote an open interval fulfilling $\emptyset \neq I \subset \mathbb{R}$.
For $f : I \to \mathbb{R}$ and $x \in I$, if the finite limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists, we call it the derivative of f at x and we say that f is differentiable [at x].

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We can also speak about differentiability from the left/right.

Approximately constant functions with small error

Proposition

Let $c \geq 0$ and $p > 1$. If $f : I \to \mathbb{R}$ satisfies the inequality

$$|f(y) - f(x)| \leq c|y - x|^p$$

for all $x, y \in I$, then f is constant.

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Hint: $f' = 0$.

Localization

Definition

We say that $f\colon I \to \mathbb{R}$ satisfies a property P locally (or shortly: f is **locally** P) if, for every $x \in I$, there exists an open interval $I(x)$ fulfilling $x \in I(x) \subset I$, such that $f|_{I(x)}$ satisfies P.

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Definition

We say that the property P is localizable (or "P satisfies the localization principle") if, for any function $f: I \to \mathbb{R}$, the following implication holds:

$$f \text{ is locally P} \quad \Rightarrow \quad f \text{ is P.}$$

Lists of (counter)examples

Localizable properties

- local concepts (continuity, differentiability);
- f is a polynomial of degree at most n $[\Leftrightarrow \quad f^{(n+1)} = 0]$;
- monotonicity (in a given sense) [!]

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Non-localizable properties

- boundedness;
- Lipschitz property;
- absolute/uniform continuity.

As an example one may consider $f(x) = 1/x \quad (x > 0)$.

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For each $u \in I$, f is increasing on a neighbourhood $I(u)$ of u.

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Let $x, y \in I$, $x < y$ and let $M(x)$ denote the set of of all $z \in]x, y]$ admitting a partition $x = x_0 < x_1 < \cdots < x_{n-1} < x_n = z$ of $[x, z]$ (with some $n \in \mathbb{N}$) such that, for each $j \in \{1, 2, \ldots, n\}$, x_{j-1} , $x_j \in I(u_j)$ with some $u_j \in I$.

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Clearly, for every $z \in M(x)$, we have

$$f(x) = f(x_0) \leq f(x_1) \leq \cdots \leq f(x_{n-1}) \leq f(x_n) = f(z) .$$

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$$f(x) = f(x_0) \leq f(x_1) \leq \cdots \leq f(x_{n-1}) \leq f(x_n) = f(z).$$

By basic arguments we can also check that

- $M(x) =]x, v[$ or $M(x) =]x, v]$ with some $v \in]x, y]$;
- $v \in M(x)$; $\quad v = y$.

Concept of convexity

Definition

We call $f\colon I \to \mathbb{R}$ convex if the inequality

$$f\big((1-t)\cdot x + t\cdot y\big) \le (1-t)\cdot f(x) + t\cdot f(y) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0,1]$.

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Proposition

If $f\colon I \to \mathbb{R}$ is convex and $x, y, z \in I$ fulfil $x < y < z$, then

$$\frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(x)}{z-x} \leq \frac{f(z)-f(y)}{z-y}.$$

Conversely, any of these inequalities (satisfied identically) implies the convexity of f.

Derivatives and convexity

For $f\colon I \to \mathbb{R}$, the following assertions are equivalent:

(A) f is convex.

(B) f is differentiable from the left/right. Furthermore, for any $x, y \in I$ with $x < y$, we have

$$f'_-(x) \le f'_+(x) \le f'_-(y) \le f'_+(y)\,.$$

(C) f is differentiable from the left/right. Furthermore, for any $x \in I$, we have

$$\sup_{x>s\in I} \frac{f(s)-f(x)}{s-x} \le f'_-(x) \le f'_+(x) \le \inf_{x<t\in I} \frac{f(t)-f(x)}{t-x}\,.$$

(D) For every $x \in I$, there exists $\lambda_x \in \mathbb{R}$ such that

$$f(y)-f(x) \ge \lambda_x(y-x) \quad \text{for all} \quad y \in I\,.$$

Related Remarks

Hints for the proof:

(A) $\Rightarrow$ (C) $\Rightarrow$ (D) $\Rightarrow$ (A); $f'_{-}(x) \leq \lambda_x \leq f'_{+}(x)$.

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(A) $\Rightarrow$ (B): Use the Proposition.

(B) $\Rightarrow$ (A): It is clear that f is continuous (from both direction). According to a generalized MVT, for every $x, y \in I$ with $x < y$, there exists $u \in]x, y[$ satisfying

$$f'_-(u) \leq \frac{f(y) - f(x)}{y - x} \leq f'_+(u).$$

We can again apply the Proposition.

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We can again apply the Proposition.

Remark:

Using property (B) and the localization principle for increasing functions, we obtain that convexity is localizable.

Approximate convexity with small error

Theorem (Z. B. and N. Nagy, 2013)

Suppose that $c \geq 0$, $p > 1$, and $f : I \to \mathbb{R}$ satisfies the inequality

$$f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y)+c\big(\lambda(1-\lambda)\,|x-y|\big)^p \quad (2)$$

for every $x, y \in I$ and $\lambda \in [0,1]$. Then f is convex (so f satisfies (2) with $c = 0$ as well).

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Hints for the proof:
We prove that f satisfies property (C).

Basic concepts

In this section, let K denote a subfield of the real number field $\mathbb{R}$ and X be a vector space over K. We denote the set of positive elements of K by K^+.

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A mapping $F : X \to \mathbb{R}$ is called

- *additive* if the equation $F(x+y) = F(x) + F(y)$ is fulfilled for every $x, y \in X$;
- *subadditive* if the inequality $F(x+y) \leq F(x) + F(y)$ is fulfilled for every $x, y \in X$;
- *K-homogeneous* if the equation $F(rx) = rF(x)$ is fulfilled for every $x \in X$ and $r \in K$;
- *positively K-homogeneous* if the equation $F(rx) = rF(x)$ holds for all $x \in X$ and $r \in K^+$;
- *K-linear* if F is additive and (positively) K-homogeneous;
- *K-sublinear* if F is subadditive and positively K-homogeneous.

Abstract tools and concepts

Lemma

Let $F : X \to \mathbb{R}$ be K-sublinear and $u \in X$ be fixed. Then there exists a K-linear mapping $\varphi : X \to \mathbb{R}$ such that $\varphi(u) = F(u)$ and $\varphi(x) \leq F(x)$ for every $x \in X$.

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Let $\mathcal{A}$ denote the set of all additive mappings $A : X \to \mathbb{R}$ and let $\mathcal{A}_K$ denote the set of all K-linear mappings $A : X \to \mathbb{R}$.
Let us note that, for instance, in the particular case $X = \mathbb{R}^N$, we have $\mathcal{A} = \mathcal{A}_{\mathbb{Q}}$.

Assumptions concerning the domain

A subset D of the space X is called *K-algebraically open* if, for every $x \in D$ and $u \in X$, there exists $\delta > 0$ such that

$$x + ru \in D \quad \text{whenever} \quad r \in K \cap]-\delta, \delta[\,.$$

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We say that D is *K-convex* if

$$rx + (1-r)y \in D \quad \text{for every} \quad x, y \in D \;\; \text{and} \;\; r \in K \cap [0,1].$$

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In this section, let D denote a K-algebraically open and K-convex subset of X. As an example, one may suppose that D is an open interval and $X = \mathbb{R}$.

K-subdifferential and radial K-derivatives

Let $f : D \to \mathbb{R}$ and $x_0 \in D$. The set

$$\partial_K f(x_0) = \{\, A \in \mathcal{A}_K \mid \forall x \in D \ : \ f(x_0) + A(x - x_0) \leq f(x) \,\}$$

is called the *K-subdifferential of f at x_0*. If $A \in \partial_K f(x_0)$, we say that A is a *K-subgradient* of the function f at the point x_0.

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Let $f : D \to \mathbb{R}^N$, $x_0 \in D$, and $u \in X$. If the limit

$$d_K f(x_0\,,\, u) = \lim_{K^+ \ni r \to 0} \frac{1}{r}\,(f(x_0 + ru) - f(x_0))$$

exists, it is called the radial K-derivative of f at x_0 in the direction u. We shall say that f is *radially K-differentiable* $\left[\text{at } x_0\right]$ if $d_K f(x_0\,,\, v) \in \mathbb{R}^N$ exists for every $v \in X$.

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Let $f : D \to \mathbb{R}$ and $x_0 \in D$. If f is radially K-differentiable at x_0, the set

$$\delta_K f(x_0) = \{\, A \in \mathcal{A}_K \mid A(v) \le d_K f(x_0\,,\, v) \text{ for every } v \in X \,\}$$

is called the *K-subderivative of f at x_0*.

Operation rules for radial K-derivatives

Theorem (Z. B. and Zs. Páles, 2006)

Let us suppose that $f : D \to \mathbb{R}^N$ is radially K-differentiable at $x_0 \in D$, $E \subset \mathbb{R}^N$ is an open set such that $f(D) \subset E$, $F : E \to \mathbb{R}$ is differentiable at $f(x_0)$, and let $h(x) = F(f(x))$ for all $x \in D$. Then $h : D \to \mathbb{R}$ is also radially K-differentiable at x_0, and, for every $v \in X$, we have

$$d_K h(x_0, v) = F'(f(x_0)) d_K f(x_0, v).$$

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$$d_K h(x_0, v) = F'(f(x_0)) d_K f(x_0, v).$$

Corollary

If $f, g : D \to \mathbb{R}$ are radially K-differentiable at $x_0 \in D$, then $f + g$ and fg are also radially K-differentiable at x_0, and, for every $v \in X$, we have

$$\begin{aligned} d_K(f+g)(x_0, v) &= d_K f(x_0, v) + d_K g(x_0, v), \\ d_K(fg)(x_0, v) &= g(x_0) d_K f(x_0, v) + f(x_0) d_K g(x_0, v). \end{aligned}$$

K-convex and Jensen-convex functions

Definition

We call $f : D \to \mathbb{R}$ *K-convex* if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \tag{3}$$

holds for every $x, y \in D$, $\lambda \in [0,1] \cap K$.

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In this terminology, *convex* functions on an interval are exactly the $\mathbb{R}$-convex ones.

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In this terminology, *convex* functions on an interval are exactly the $\mathbb{R}$-convex ones.

Definition

We call $f : D \to \mathbb{R}$ *Jensen-convex* if

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} \tag{4}$$

holds for every $x, y \in D$.

Jensen-convex functions on an interval

Theorem (Jensen, 1906)

Let $X = \mathbb{R}$ and D be an open interval.
$f : D \to \mathbb{R}$ is Jensen-convex $\iff$ f is $\mathbb{Q}$-convex.

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Corollary

Let $X = \mathbb{R}$ and D be an open interval.
$f : I \to \mathbb{R}$ is convex $\iff$ f is continuous and Jensen-convex.

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Thus the concept of K-convex functions involves, among others,

- Jensen-convex functions on an open interval ($K = \mathbb{Q}$);
- convex functions defined on a convex and $\mathbb{R}$-algebraically open subset of an arbitrary real linear space ($K = \mathbb{R}$).

Differentiation of K-convex functions

Theorem (Z. B. and Zs. Páles, 2006)

If $f : D \to \mathbb{R}$ is K-convex, then it is also radially K-differentiable. Moreover, for every $x \in D$, the mapping $\psi(v) = d_K f(x, v)$ $(v \in X)$ is K-sublinear, $\partial_K f(x) = \delta_K f(x)$, and, for each $u \in X$, there exists $A \in \delta_K f(x)$ such that $A(u) = d_K f(x, u)$.

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Proposition

If $f : D \to \mathbb{R}$ such that $\partial_K f(x) \neq \emptyset$ for every $x \in D$, then f is K-convex.

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Theorem (Z. B. and Zs. Páles, 2006)

Suppose that $g, h : D \to \mathbb{R}$ are K-convex functions and $\partial_K g(x) = \partial_K h(x)$ for every $x \in D$. Then there exists a constant $c \in \mathbb{R}$ such that $g(x) = h(x) + c$ for all $x \in D$.

The difference operator

Let I denote a non-empty open interval in $\mathbb{R}$.
For $f : I \to \mathbb{R}$, $x \in I$ and $h \in \mathbb{R}$ fulfilling $x + h \in I$, let

$$\Delta_h f(x) = f(x+h) - f(x).$$

Higher order iterates of the difference operator are defined by the recursion

$$\Delta_h^{n+1} f(x) = \Delta_h(\Delta_h^n f(x)) \quad (n \in \mathbb{N}).$$

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$$\Delta_h^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+kh).$$

Generalized polynomials

Theorem (R. Ger, 1994; ...)

Let $n \in \mathbb{N}$. A function $f : I \to \mathbb{R}$ satisfies the functional equation

$$\Delta_h^{n+1} f(x) = 0 \quad (x \in I,\ x + (n+1)h \in I) \tag{5}$$

if, and only if, f admits a unique extension $F : \mathbb{R} \to \mathbb{R}$ of the form

$$F(x) = h_0 + \sum_{k=1}^{n} h_k(x) \ \ (x \in \mathbb{R}), \tag{6}$$

where $h_0 \in \mathbb{R}$ and, for each $k \in \{1, 2, \dots, n\}$, there exists a symmetric, k-additive function $A_k : \mathbb{R}^k \to \mathbb{R}$ such that $h_k(x) = A_k(x, x, \dots, x) \quad (x \in \mathbb{R})$.

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if, and only if, f admits a unique extension $F: \mathbb{R} \to \mathbb{R}$ of the form

$$F(x) = h_0 + \sum_{k=1}^{n} h_k(x) \ \ (x \in \mathbb{R}), \tag{6}$$

where $h_0 \in \mathbb{R}$ and, for each $k \in \{1, 2, \ldots, n\}$, there exists a symmetric, k-additive function $A_k: \mathbb{R}^k \to \mathbb{R}$ such that $h_k(x) = A_k(x, x, \ldots, x) \quad (x \in \mathbb{R})$.

Functions given by the representation (6) are called *generalized polynomials of degree at most n*, while the functional equation (5) was considered by Fréchet (in 1909).

Generalized monomials

The function h_n is called *generalized monomial of degree n*, and it represents the general solution of the functional equation

$$\Delta_y^n h(x) = n!h(y) \quad (x, y \in \mathbb{R}). \tag{7}$$

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Let us note that a generalized monomial of degree n is

- additive if $n = 1$;
- quadratic (solution of the norm square equation) if $n = 2$.

Dinghas' interval derivative

A. Dinghas introduced (in 1966) the n-th interval derivative of $f : I \to \mathbb{R}$ at $x \in I$ by the expression

$$D^n f(x) = \lim_{\substack{a \le x \le b \\ b-a \to 0}} \left(\frac{-n}{b-a} \right)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} f\left(\left(1 - \frac{k}{n}\right) a + \frac{k}{n} b \right)$$

whenever the limit exists.

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Let us note that the finite limit

$$D^1 f(x) = \lim_{\substack{a \le x \le b \\ b-a \to 0}} \frac{f(b) - f(a)}{b-a}$$

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In the higher order cases, the situation is different.

Characterization of generalized polynomials

Theorem (A. Simon and P. Volkmann, 1994)

Let $n \in \mathbb{N}$ and $f : I \to \mathbb{R}$. The following assertions are equivalent:

(A) $\Delta_h^n f(x) = 0$ for all $x \in I$ and $h \in \mathbb{R}$ fulfilling $x + nh \in I$.

(B) $D^n f(x) = 0$ for every $x \in I$.

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(B) $D^n f(x) = 0$ for every $x \in I$.

A. Gilányi considered (in 1997) the analogous limit

$$\tilde{D}^n f(x) = \lim_{\substack{y \le x \le y+nh \\ h \to 0}} \frac{\Delta_h^n f(y) - n! f(h)}{h^n}$$

and proved, for any $n \in \mathbb{N}$ and $f : \mathbb{R} \to \mathbb{R}$, that

$$\tilde{D}^n f(x) = 0 \quad (x \in \mathbb{R}) \quad \Leftrightarrow \quad \Delta_u^n f(x) = n! f(u) \quad (x, u \in \mathbb{R}).$$

Generalized Jensen-convexity of higher order

A local characterization of Jensen-convexity of higher order was elaborated by A. Gilányi and Zs. Páles (in 2001) in a somewhat more general context.

T-convexity

Let $T = (t_1, \ldots, t_{n+1})$, where $t_1, \ldots, t_{n+1}$ are fixed positive numbers. For $f : I \to \mathbb{R}$, $x \in I$ and $h > 0$ such that $x + (t_1 + \cdots + t_{n+1})h \in I$, let

$$\Delta_h^T f(x) := \Delta_{t_1 h} \ldots \Delta_{t_{n+1} h} f(x) .$$

We say that $f : I \to \mathbb{R}$ is T-convex if $\Delta_h^T f(x) \geq 0$ for all $x \in I$, $h > 0$ such that $x + (t_1 + \cdots + t_{n+1})h \in I$.

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Clearly, T-convexity and cT-convexity are equivalent for $c>0$. In the case $t_1=\dots=t_{n+1}=1$ the notion of T-convexity is obviously the same as Jensen-convexity of order n.

Lower Dinghas type interval derivatives

Lower T-Dinghas interval derivative of f

The lower T-Dinghas interval derivative of $f : I \to \mathbb{R}$ at $\xi \in I$ is defined by

$$\underline{D}^T f(\xi) := \liminf_{\substack{(x,h)\to(\xi,0)\\ x\le\xi\le x+(t_1+\cdots+t_{n+1})h}} \frac{\Delta_h^T f(x)}{(t_1 h)\dots(t_{n+1} h)}.$$

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n-th order lower Dinghas interval derivative of f

Accordingly, if n denotes a positive integer, the n-th order lower Dinghas interval derivative of $f : I \to \mathbb{R}$ at $\xi \in I$ is defined by

$$\underline{D}^n f(\xi) := \liminf_{\substack{(x,h)\to(\xi,0)\\ x\le\xi\le x+nh}} \frac{\Delta_h^n f(x)}{h^n}.$$

Localization of T-convexity

$$\underline{D}^T f(\xi) := \liminf_{\substack{(x\,,\,h)\to(\xi\,,\,0)\\ x\le\xi\le x+(t_1+\cdots+t_{n+1})h}} \frac{\Delta_h^T f(x)}{(t_1 h)\ldots(t_{n+1}h)}\,, \quad \underline{D}^n f(\xi) := \liminf_{\substack{(x\,,\,h)\to(\xi\,,\,0)\\ x\le\xi\le x+nh}} \frac{\Delta_h^n f(x)}{h^n}$$

Gilányi and Páles proved a strong connection between the above two concepts. Namely, they established that a function $f : I \to \mathbb{R}$ is T-convex if, and only if, $\underline{D}^T f(\xi) \ge 0$ for every $\xi \in I$. Considering the particular case when $t_1 = \cdots = t_{n+1} = 1$, one obtains the following statement:

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Theorem (A. Gilányi and Zs. Páles, 2001)

A function $f : I \to \mathbb{R}$ is Jensen-convex of order n if, and only if, $\underline{D}^{n+1} f(\xi) \geq 0$ for every $\xi \in I$.

Difference operators

As well as in the previous section, I denotes an open subinterval of the real line.

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We may define n-th order differences for $f\colon I \to \mathbb{R}$, $x \in I$, $1 < n \in \mathbb{N}$, and $u_j \in \mathbb{R}$ $(j = 1, 2, \ldots, n)$ fulfilling

$$x + \sum_{j=1}^{n} \varepsilon_j u_j \in I \quad \text{for every} \quad \varepsilon_j \in \{0, 1\} \quad (j = 1, 2, \ldots, n),$$

by the recursion

$$\Delta_{u_1, \ldots, u_{n-1}, u_n} f(x) = \Delta_{u_n} \Delta_{u_1, \ldots, u_{n-1}} f(x) \, .$$

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$$\Delta_{u_1, \ldots, u_{n-1}, u_n} f(x) = \Delta_{u_n} \Delta_{u_1, \ldots, u_{n-1}} f(x) .$$

Let us note that

$$\Delta_{u_1, u_2} f(x) = \Delta_{u_2} \Delta_{u_1} f(x) = \Delta_{u_1} \Delta_{u_2} f(x)$$

whenever $\{ x , x + u_1 , x + u_2 , x + u_1 + u_2 \} \subset I$.

Strong K-differentiability

In this part of the presentation, $p \in \mathbb{N}$, K denotes a subfield of $\mathbb{R}$ and $K_+ = K \cap]0, +\infty[$.

Definition

Let $f : I \to \mathbb{R}$, $x_0 \in I$, and $\mathbf{h} = (h_1, \ldots, h_p) \in \mathbb{R}^p$. If the finite limit

$$D_K^p f(x_0 ; \mathbf{h}) = \lim_{\substack{x \to x_0 \\ K_+^p \ni \mathbf{r} \to \mathbf{0}}} \frac{1}{r_1 r_2 \cdots r_p} \Delta_{r_1 h_1, r_2 h_2, \ldots, r_p h_p} f(x)$$

(where $\mathbf{r} = (r_1, \ldots, r_p)$) exists, it is called the pth order strong K-derivative of f at x_0 in the direction $\mathbf{h} = (h_1, \ldots, h_p)$, and we shall say that f is *strongly K-differentiable of order p [at the point x_0 in the direction* $\mathbf{h} = (h_1, \ldots, h_p)$].

Smooth example

Let

$$\mathcal{S}_K^p(I) = \{\, f : I \to \mathbb{R} \,|\, f \text{ is strongly } K\text{-differentiable of order } p \,\}.$$

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$$\mathcal{C}^p(I) = \{\, f : I \to \mathbb{R} \,|\, f \text{ is } p \text{ times continuously differentiable} \,\}.$$

Example

If $f \in \mathcal{C}^p(I)$, then $f \in \mathcal{S}_K^p(I)$. Moreover, for every $x_0 \in I$ and $\mathbf{h} = (h_1\,,\,\ldots\,,\,h_p) \in \mathbb{R}^p$,

$$D_K^p f(x_0\,;\,\mathbf{h}) = f^{(p)}(x_0)h_1 h_2 \cdots h_p\,.$$

Algebraic example

Example

Suppose that $F_0 \in \mathbb{R}$, $F_k : \mathbb{R}^k \to \mathbb{R}$ is symmetric and multi-additive $(k = 1, 2, \dots, p-1)$, $F_p : \mathbb{R}^p \to \mathbb{R}$ is symmetric and multi-linear over K, and let $f_0(x) = F_0$ for all $x \in \mathbb{R}$,

$$f_k(x) = F_k(\overbrace{x, x, \dots, x}^{k}) \quad \text{for every} \quad x \in \mathbb{R}, \; k = 1, 2, \dots, p,$$

and

$$f(x) = \sum_{k=0}^{p} f_k(x) \qquad (x \in I).$$

Then $f \in \mathcal{S}_K^p(I)$. Moreover, for every $x_0 \in I$ and $\mathbf{h} = (h_1, \dots, h_p) \in \mathbb{R}^p$,

$$D_K^p f(x_0; \mathbf{h}) = p! F_p(h_1, \dots, h_p).$$

Decomposition of strongly K-differentiable functions

Theorem (Z. B.)

Let $f : I \to \mathbb{R}$. Then $f \in \mathcal{S}^p_K(I)$ if, and only if, there exist $g \in \mathcal{C}^p(I)$ and multi-additive functions $F_k : \mathbb{R}^k \to \mathbb{R}$ $(k = 1, \ldots, p)$ such that F_p is multi-linear over K and

$$f(x) = g(x) + \sum_{k=1}^{p} F_k(\overbrace{x, x, \ldots, x}^{k}) \quad \text{for every} \quad x \in I\,.$$

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Using this decomposition theorem, we can prove the localization principle for such decompositions.

Localization principle for the decomposition

Theorem (Z. B.)

Suppose that $f : I \to \mathbb{R}$ such that for every $x \in I$ there exist an open interval $J_x \subset I$ fulfilling $x \in J_x$, a p times continuously differentiable function $g_x : J_x \to \mathbb{R}$, and a generalized polynomial $P_x : \mathbb{R} \to \mathbb{R}$ of order at most p satisfying

$$f(y) = g_x(y) + P_x(y) \quad \text{for all} \quad y \in J_x \,.$$

Then there exist a p times continuously differentiable function $g : I \to \mathbb{R}$ and a generalized polynomial $P : \mathbb{R} \to \mathbb{R}$ of order at most p satisfying

$$f(x) = g(x) + P(x) \quad \text{for all} \quad x \in I \,.$$

Strong geometric derivatives

Let $0 < \theta < 1$ and $f : I \to \mathbb{R}$. For each $h \in \mathbb{R}$ and $x \in I$, the extended real numbers

$$\underline{D}_h^{\theta} f(x) = \liminf_{\substack{y \to x \\ n \to \infty}} \frac{f(y + \theta^n h) - f(y)}{\theta^n}$$

and

$$\overline{D}_h^{\theta} f(x) = \limsup_{\substack{y \to x \\ n \to \infty}} \frac{f(y + \theta^n h) - f(y)}{\theta^n}$$

will be called the lower/upper strong θ-derivatives of f at x in the direction h.

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Clearly,

$$\underline{D}_h^{[\theta]} f(x) \leq \overline{D}_h^{[\theta]} f(x).$$

Strong geometric differentiability

Definition. Let $0 < \theta < 1$, $f : I \to \mathbb{R}$, $x_0 \in I$, $h \in \mathbb{R}$. If

$$-\infty < \underline{D}_h^{[\theta]} f(x_0) = \overline{D}_h^{[\theta]} f(x_0) < +\infty \,,$$

then f is called *strongly θ-differentiable*, and

$$D_h^{[\theta]} f(x_0) = \underline{D}_h^{[\theta]} f(x_0)$$

is called the *strong θ-derivative* of f

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Remark. Let us note that

$$D_h^{[\theta]} f(x_0) = \lim_{\substack{x \to x_0 \\ n \to \infty}} \frac{f(x + \theta^n h) - f(x)}{\theta^n} .$$

Derivatives of convex functions

Theorem

Let $0 < \theta < 1$. If $f : I \to \mathbb{R}$ is convex, x_1 , $x_2 \in I$ such that $x_1 < x_2$, and $h > 0$, then

$$-\infty < \overline{D}_h^{[\theta]} f(x_1) \leq \underline{D}_h^{[\theta]} f(x_2) < +\infty \,. \tag{8}$$

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$$-\infty < \overline{D}_h^{[\theta]} f(x_1) \le \underline{D}_h^{[\theta]} f(x_2) < +\infty . \tag{8}$$

Let $\mathcal{K}_+^{[\theta]}(I)$ denote the set of functions $f : I \to \mathbb{R}$ that fulfil inequality (8) for all $h > 0$ and x_1 , $x_2 \in I$ such that $x_1 < x_2$.

Dyadic derivatives

In the rest of this talk we restrict our considerations to the case $\theta = \frac{1}{2}$. For simplicity, we shall write $\diamond$ in place of $[1/2]$ as an upper index.

Accordingly, instead of strong $\frac{1}{2}$-derivatives, we write strong *dyadic* derivatives, while instead of strongly $\frac{1}{2}$-differentiable functions, we write strongly dyadically differentiable functions.

Decomposition Theorems

Theorem (Z. B.)

Let us suppose that the function $f : I \to \mathbb{R}$ belongs to the class $\mathcal{K}_+^\diamond(I)$. Then there exist a convex function $g : I \to \mathbb{R}$ and an additive mapping $A : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = g(x) + A(x) \quad \text{for every} \quad x \in I\,.$$

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Theorem (Z. B., 2001; D. Broszka and Z. Grande, 2007)

A function $f : I \to \mathbb{R}$ is strongly dyadically differentiable if, and only if, there exist a continuously differentiable function $g : I \to \mathbb{R}$ and an additive mapping $A : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = g(x) + A(x) \quad \text{for every} \quad x \in I\,.$$

Approximately Wright-convex functions

In this section, let $0 \leq \varepsilon \in \mathbb{R}$ and $1 < p \in \mathbb{R}$.

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Theorem (Z. B.)

If $f : I \to \mathbb{R}$ satisfies the inequality

$$\begin{aligned} f\big(\lambda x + (1-\lambda)y\big) + f\big((1-\lambda)x + \lambda y\big) \\ \le f(x) + f(y) + \varepsilon\big(\lambda(1-\lambda)|x-y|\big)^p \end{aligned} \tag{9}$$

for every $x, y \in I$ and $\lambda \in [0,1]$, then $f \in \mathcal{K}_+^\diamond(I)$.

Approximately Wright-convex functions

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for every $x, y \in I$ and $\lambda \in [0,1]$, then $f \in \mathcal{K}_{+}^{\diamond}(I)$.

Corollary

If $f : I \to \mathbb{R}$ satisfies the inequality (9) for every $x, y \in I$ and $\lambda \in [0,1]$, then f is Wright-convex (i.e., f satisfies the inequality (9) with $\varepsilon = 0$ as well).

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Thank you for your kind attention!