Faculty of Mathematics and Computer Science
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# On some properties of functions connected with focal points of entropy

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$\mathcal{F} \subset \exp(X) \setminus \emptyset$ - a family such that each nonempty open set contains some element of $\mathcal{F}$.

$Y$- open subset of $X$.
$\vartheta^Y_{\mathcal{F}} = \{(A_1, \dots A_m) : m \in \mathbb{N}, A_1 \dots A_m \in \mathcal{F}, A_1, \dots A_m \subset Y, \overline{A_i} \cap \overline{A_j} = \emptyset\}$.
For a sequence $A = (A_1, \dots, A_m) \in \vartheta^X_{\mathcal{F}}$ we define the matrix $\mathcal{M}_{A,f} = [m_{ij}]_{i,j=1}^m$ in the following way:

$$m_{ij} = \begin{cases} 1 \text{ if } A_i \underset{f}{\rightarrow} A_j \\ 0 \text{ otherwise.} \end{cases}$$

**The entropy of a function $f$ with respect to a sequence $A = (A_1, \dots, A_m) \in \vartheta^X_{\mathcal{F}}$:**

$$H_f(F) = \begin{cases} \log \sigma(\mathcal{M}_{F,f}) & \text{if } \sigma(\mathcal{M}_{F,f}) > 0, \\ 0 & \text{if } \sigma(\mathcal{M}_{F,f}) = 0. \end{cases}$$

where $\sigma(\mathcal{M}_{F,f}) = \limsup_{n \to \infty} \sqrt[n]{\operatorname{tr}(\mathcal{M}^n_{F,f})}$.

$$H_{\mathcal{F},f}(Y) = \sup\left\{\frac{1}{n}H_{f^n}(F) : F \in \vartheta_{\mathcal{F}}^{Y} \wedge n \in \mathbb{N}\right\}.$$

$$d(\mathcal{F}, f, Y) = \begin{cases} \frac{H_{\mathcal{F},f}(Y)}{h(f)} & \text{if } h(f) \in (0,\infty), \\ 1 & \text{if } H_{\mathcal{F},f}(Y) = \infty \text{ or } h(f) = 0, \\ 0 & \text{if } H_{\mathcal{F},f}(Y) \in [0,\infty) \text{ and } h(f) = \infty. \end{cases}$$

**A density of entropy of $f$ with respect to $\mathcal{F}$ at the point $x_0$**:

$$E_{\mathcal{F},f}(x_0) = \inf\{d(\mathcal{F}, f, V) : V \in O(x_0)\}$$

We say that a point $x_0 \in X$ is an **$\mathcal{F}$-focal point of entropy of $f$** if:

$$E_{\mathcal{F},f}(x_0) = 1.$$

$E_{\mathcal{F}}(f)$ - the set of all $\mathcal{F}$-focal points of entropy of $f$.

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A point $x_0 \in X$ is called a **recurrent point** for $f$ if there exists a sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integers such that $x_0 = \lim_{k\to\infty} f^{n_k}(x_0)$.

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$$R(f) \subset \Omega(f)$$

**Theorem 1.**

*Let $f : X \to X$ be a function such that $h(f) > 0$ and $\mathcal{F} \subset \exp(X) \setminus \{\emptyset\}$ be a family such that each open set contains some element of $\mathcal{F}$. If $x \in X$ is an $\mathcal{F}$-focal point of entropy of $f$ then $x$ is a nonwandering point for $f$.*

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$h(f) = 0 \Rightarrow E_{\mathcal{F}}(f) = X$.

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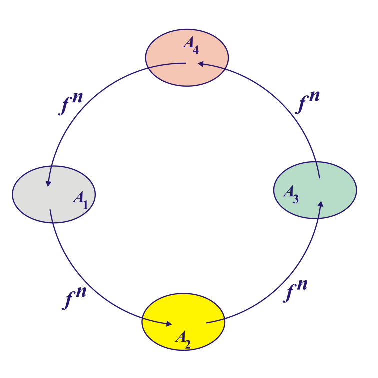
Let $U \in O(x_0)$ be such that $1 \notin U$. For any $i \in \mathbb{N}$ we have $U \cap f^i(U) = U \cap \{1\} = \emptyset$.

We will say that $f \in \mathfrak{J}(\mathcal{F})$ if for any $n \in \mathbb{N}$ and for any sequence $(A_1, ..., A_{m-1}, A_1) \in \vartheta_{\mathcal{F}}^X$ such that

$$A_1 \underset{f^n}{\rightarrow} A_2 \underset{f^n}{\rightarrow} ... \underset{f^n}{\rightarrow} A_{m-1} \underset{f^n}{\rightarrow} A_m = A_1$$

there exists $x_0 \in A_1$ such that $f^{n+m-1}(x_0) = x_0$ and $f^{n+i-1}(x_0) \in A_{i+1}$, for $i \in \{1, 2, ..., m-1\}$.



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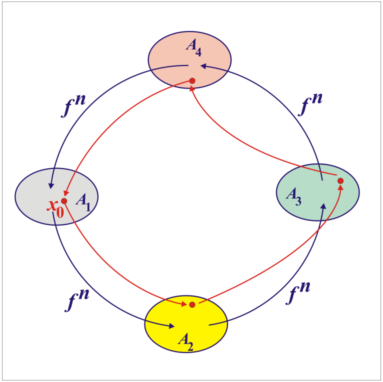

**Theorem 2.**

*Let $f : X \to X$ be a function such that $h(f) > 0$ and $f \in J(\mathcal{F})$. If $x_0$ is $\mathcal{F}$-focal point of entropy of $f$, then $h(f \restriction (U \cap R(f))) = h(f)$ for any $U \in O(x_0)$.*

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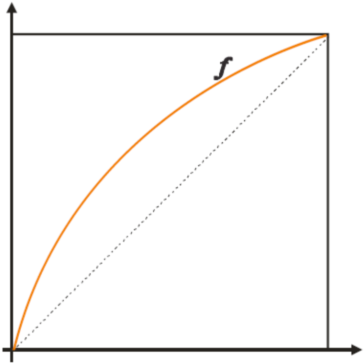

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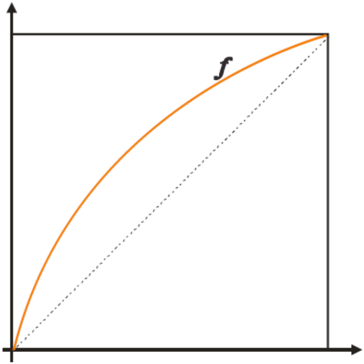


$$E_{\mathcal{F}}(f) = X \text{ but } R(f) = \{0, 1\}.$$

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*A topological space $X$ is called an $m$-dimensional manifold with boundary if it is a second countable Hausdorff space and every point $x \in X$ has a neighbourhood that is homeomorfic to an open subset of $\mathbb{H}^m = \{(x_1, \dots, x_m) : x_m \geq 0\}$.*

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We say that a continuous function $f : X \to X$ is **$\mathcal{F}$-modifiable at a point $x_0$** if

for any $\varepsilon > 0$ there exists a continuous function $g$ such that

$x_0$ is an $\mathcal{F}$-focal entropy point of $g$, $\rho_u(f, g) < \varepsilon$ and $\nabla(f, g) \subset B_\rho(x_0, \varepsilon)$.

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**Theorem 4.**

*Let $X$ be a compact metric space being an $m$-dimensional manifold, $f : X \to X$ be a continuous function, $x_0 \in \mathrm{Fix}(f)$ and $\mathcal{F}$ be a family of all closed subsets of $X$. Then $f$ is $\mathcal{F}$-modifiable at the point $x_0$.*

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