Artur Bartoszewicz

Large free sets in powers of universal algebras

Joint results with Taras Banakh and Szymon Głąb

Background

The classical Fichtenholz-Kantorovich-Hausdorff Theorem says that the power-set $\mathcal{P}(X)$ of any infinite set X contains an independent family $\mathcal{I} \subset \mathcal{P}(X)$ of cardinality $|\mathcal{I}| = |\mathcal{P}(X)|$. The independence of $\mathcal{I}$ means that for any disjoint finite subsets $\mathcal{A}, \mathcal{B} \subset \mathcal{I}$ the intersection $(\bigcap_{A\in\mathcal{A}} A) \cap (\bigcap_{B\in\mathcal{B}} X \setminus B)$ is not empty.

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Introduction

By an *algebraic operation* on a set A we understand a function $\alpha : A^{n_\alpha} \to A$ defined on a finite power of the set A. The number $n_\alpha \in \omega$ is called the *arity* of the algebraic operation α.

Let $(A, \mathcal{A})$ be a *universal algebra*, i.e., a set A endowed with a family of algebraic operations $\mathcal{A}$.

A subset $S \subset A$ is called a *subalgebra* of the algebra $(A, \mathcal{A})$ if $\alpha(S^{n_\alpha}) \subset S$ for each algebraic operation $\alpha \in \mathcal{A}$. In this case S is a universal algebra endowed with the family $\mathcal{A}|S = \{\alpha|S^{n_\alpha}\}_{\alpha \in \mathcal{A}}$ of restricted algebraic operations.

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A function $h : S \to A$ is called a *homomorphism* if it preserves algebraic operations in the sense that $h(\alpha(x_1, \ldots, x_{n_\alpha})) = \alpha(h(x_1), \ldots, h(x_{n_\alpha}))$ for any algebraic operation $\alpha \in \mathcal{A}$ and points $x_1, \ldots, x_{n_\alpha} \in S$.

Definition

Each subset $B \subset A$ is contained in the smallest subalgebra $\bar{B} \subset A$ called *the subalgebra generated by B*.
We shall say that the subalgebra $\bar{B}$ is *freely generated by B* (or else that the subset B is *free* in A) if every function $f : B \to A$ can be extended to a homomorphism $\bar{f} : \bar{B} \to A$.

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A family $\mathcal{A}$ of operations on a set A is called

- *unital* if $\mathcal{A}$ contains the identity operation $\mathrm{id}_A : A^1 \to A$, $\mathrm{id}_A : (x_1) \mapsto x_1$;
- *stable under substitutions* (briefly, *substitution-stable*) if for any algebraic operation $\alpha \in \mathcal{A}$ and a function $s : \{1, \dots, n_\alpha\} \to \{1, \dots, m\}$ the algebraic operation $\alpha \circ A^s : A^m \to A$, $\alpha \circ A^s : (x_1, \dots, x_m) \mapsto \alpha(x_{s(1)}, \dots, x_{s(n_\alpha)})$, belongs to $\mathcal{A}$;
- *stable under compositions* if for any algebraic operations $\beta \in \mathcal{A}$ and $\alpha_1, \dots, \alpha_{n_\beta} \in \mathcal{A}$ of the same arity $n = n_{\alpha_i}$, $i \le n_\beta$, the algebraic operation $\beta(\alpha_1, \dots, \alpha_{n_\beta}) : A^n \to A$, $\beta(\alpha_1, \dots, \alpha_{n_\beta}) : (x_1, \dots, x_n) \mapsto \beta(\alpha_1(x_1, \dots, x_n), \dots, \alpha_{n_\beta}(x_1, \dots, x_n))$ belongs to $\mathcal{A}$;
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Main results

Let $(A, \mathcal{A})$ be a universal algebra and $\bar{\mathcal{A}}$ be the smallest clone containing the operation family $\mathcal{A}$.

Observation

It is standard to prove that for each subset $B \subset A$ the subalgebra $\bar{B}$ generated by B coincides with the set $\bar{\mathcal{A}}(B) = \bigcup_{\alpha \in \bar{\mathcal{A}}} \alpha(B^{n_\alpha})$. This means that for each point $y \in \bar{B}$ we can find an algebraic operation $\alpha \in \bar{\mathcal{A}}$ and points $x_1, \ldots, x_{n_\alpha} \in B$ such that $y = \alpha(x_1, \ldots, x_n)$.
We can additionally assume that the points $x_1, \ldots, x_{n_\alpha}$ are pairwise distinct.

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Lemma

A subset $B \subset A$ of a universal algebra $(A, \mathcal{A})$ is free if and only if for any distinct algebraic operations $\alpha, \beta \in \bar{\mathcal{A}}$ of the same arity $n = n_\alpha = n_\beta$ the inequality $\alpha(x_1, \ldots, x_n) \neq \beta(x_1, \ldots, x_n)$ holds for any pairwise distinct points $x_1, \ldots, x_n \in B$.

Main results

Theorem 1

For any universal algebra $(A, \mathcal{A})$ of cardinality $|A| \geq 2$ and any infinite set X of cardinality $|X| \geq |\mathcal{A}|$, the universal algebra $(A^X, \mathcal{A}^X)$ contains a free subset $B \subset A^X$ of cardinality $|B| \geq 2^{|X|}$.

The proof of this theorem uses the Fichtenholz-Kantorovich-Hausdorff Theorem, which in its turns, can be considered as a special case of our result.

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Corollary

For any infinite set X the Boolean algebra $\mathcal{P}(X)$ contains an independent subset $\mathcal{I} \subset \mathcal{P}(X)$ of cardinality $|\mathcal{I}| = |\mathcal{P}(X)|$.

Proof

Using characteristic functions, identify the Boolean algebra $\mathcal{P}(X)$ with the X-th power $\mathbf{2}^X$ of a two-element Boolean algebra $\mathbf{2}$. By Theorem 1, the Boolean algebra $\mathcal{P}(X)$ contains a free subset $\mathcal{I} \subset \mathcal{P}(X)$ of cardinality $|\mathcal{I}| = |\mathcal{P}(X)|$. It is easy to check that the family $\mathcal{I}$ is independent.

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Remark

In light of the Fichtenholz-Kantorovich-Hausdorff Theorem it is interesting to remark that in certain models of ZFC the smallest cardinality $\mathfrak{i}$ of a maximal independent subset in the Boolean algebra $\mathcal{P}(\omega)$ is strictly smaller than the cardinality of continuum 2^{ω}. This fact witnesses that Theorem 1 cannot be proved by a maximality argument using Zorn's Lemma.

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Thank you for your attention.