Recent results on the conjugacy equation on intervals

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A homeomorphism (respectively, continuous surjection) φ satisfying the functional equation

$$\varphi \circ f = g \circ \varphi, \tag{1}$$

where f and g are given interval self-maps, is said to be a *topological conjugacy* (respectively, *topological semi-conjugacy*) between f and g.

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A continuous map $f : I \to I$ is said to be *n-modal* ($n \geq 1$) if there exists a partition $(t_j)_{j=0}^{n+1}$ of the interval I such that for every $j \in \{1, \dots, n+1\}$, $f|_{[t_{j-1}, t_j]}$ is strictly monotone and $[t_{j-1}, t_j]$ is the maximal closed interval with this property.

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An n-modal map $f : I \to I$ is called *piecewise expanding* if there exists a $\mu > 1$ such that

$$|f(x) - f(y)| \geq \mu|x - y|, \qquad x, y \in (t_{j-1}, t_j),\ j \in \{1, \ldots, n+1\}.$$

Two piecewise expanding maps $f, g : I \to I$ with partitions $(t_j)_{j=0}^{n+1}$ and $(s_j)_{j=0}^{n+1}$ of the interval I, respectively, are said to be *combinatorially equivalent* if there exists a homeomorphism $h : I \to I$ such that

$$h(f^m(t_j)) = g^m(s_j), \qquad j \in \{0, \dots, n+1\},\ m \in \mathbb{N}.$$

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Theorem 1

If $f, g : [-1, 1] \to [-1, 1]$ are combinatorially equivalent piecewise expanding maps, then they are topologically conjugate.

A continuous function $f : I \to I$ is said to be a *horseshoe map* (respectively, *weak horseshoe map*) if there exists a partition $(t_j)_{j=0}^n$ ($n \geq 2$) of the interval I such that for every $j \in \{1, \ldots, n\}$, $f_{|[t_{j-1}, t_j]}$ is a homeomorphism (respectively, monotone surjection) of the interval $[t_{j-1}, t_j]$ (which is called a *lap* of f) onto I.

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We say that two horseshoe maps $f : I \to I$ and $g : J \to J$ (here and subsequently, $J := [c, d]$ with some $c, d \in \mathbb{R}$ such that $c < d$) having the same number of laps are of the *same type* if f and g are of the same type of monotonicity on their leftmost laps.

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Given an increasing function $\gamma : [0, \infty) \to [0, \infty)$ such that

$$\lim_{k\to\infty} \gamma^k(t) = 0, \qquad t \in (0, \infty),$$

we say that $T : X \to X$ is γ-*contractive* if

$$d(T(x), T(y)) \leq \gamma(d(x, y)), \qquad x, y \in X.$$

Given two horseshoe maps $f : I \to I$ and $g : J \to J$ having laps

$$[t_{j-1}, t_j], \quad [s_{j-1}, s_j], \qquad j \in \{1, \dots, n\},$$

respectively, for every $j \in \{0, \dots, n-1\}$ put

$$I_j := [t_j, t_{j+1}], \quad J_j := [s_j, s_{j+1}] \qquad \text{and} \qquad f_j := f_{|I_j}, \quad g_j := g_{|J_j}.$$

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The horseshoe map f is said to be *piecewise weakly expanding* (respectively, *piecewise γ-expanding*) if for every $j \in \{0, \ldots, n-1\}$, f_j^{-1} is weakly contractive (respectively, γ-contractive for a $\gamma : [0, \infty) \to [0, \infty)$).

K. Ciepliński, M.C. Zdun, *On uniqueness of conjugacy of continuous and piecewise monotone functions*, Fixed Point Theory Appl. 2009, Art. ID 230414, 11 pp.

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Proposition 1

Let $f : I \to I$ and $g : J \to J$ be horseshoe maps having n laps $[t_{j-1}, t_j]$ and $[s_{j-1}, s_j]$, respectively. Assume also that $\varphi : I \to J$ is a monotone and surjective solution of equation (1)*.*

(i) *If φ is increasing, then $\varphi(t_j) = s_j$ for $j \in \{0, \ldots, n\}$ and $\varphi[I_j] = J_j$ for $j \in \{0, \ldots, n-1\}$.*

(ii) *If φ is decreasing, then $\varphi(t_j) = s_{n-j}$ for $j \in \{0, \ldots, n\}$ and $\varphi[I_j] = J_{n-j-1}$ for $j \in \{0, \ldots, n-1\}$.*

Proposition 2

Assume that

(H) $f : I \to I$ *and* $g : J \to J$ *are horseshoe maps of the same type and having* n *laps* $[t_{j-1}, t_j]$ *and* $[s_{j-1}, s_j]$, *respectively,*

and g *is piecewise weakly expanding. If* $\varphi : I \to J$ *is a continuous and non-constant solution of equation (1), then* φ *is a topological semi-conjugacy. If, moreover,* n *is even and* φ *is injective, then* φ *is an increasing topological conjugacy.*

Theorem 2

If assumption (H) holds and g is piecewise γ-expanding, then there exists a unique function $\varphi : I \to J$ satisfying equation (1) and the condition

$$\varphi[I_j] \subset J_j, \qquad j \in \{0, \dots, n-1\}. \tag{2}$$

This function is an increasing topological semi-conjugacy. If, moreover, f is piecewise weakly expanding, then φ is a topological conjugacy.

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Theorem 3

If n is odd, assumption (H) holds and g is piecewise γ-expanding, then there exists a unique function $\varphi : I \to J$ satisfying equation (1) and the condition

$$\varphi[I_j] \subset J_{n-j-1}, \qquad j \in \{0, \ldots, n-1\}. \tag{3}$$

This function is a decreasing topological semi-conjugacy. If, moreover, f is piecewise weakly expanding, then φ is a topological conjugacy.

Corollary 1

If assumption (H) holds, g is piecewise γ-expanding and n is odd, then equation (1) has exactly two monotone and surjective solutions. One of them is increasing, while the other is decreasing.

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Theorem 4

If assumption (H) holds, and f and g are piecewise weakly expanding, then there exists a unique function $\varphi : I \to J$ satisfying equation (1) and condition (2). This function is an increasing topological conjugacy. If, moreover, n is odd, then there is also exactly one mapping $\varphi : I \to J$ fulfilling equation (1) and condition (3). This mapping is a decreasing topological conjugacy.

Remark 1

If assumption (H) holds, g is piecewise weakly expanding and n is even, then there is no topological conjugacy between f and g satisfying condition (3).

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Corollary 2

Let assumption (H) hold, and f and g be piecewise weakly expanding.

(i) *If n is even, then there is a unique topological conjugacy between f and g. This conjugacy is increasing.*

(ii) *If n is odd, then there are exactly two topological conjugacies between f and g. One of them is increasing, while the other is decreasing.*

For any integer $n \geq 2$ denote by $T_n, T_{n^-} : [0,1] \to [0,1]$ the piecewise monotone and linear (with slope $\pm n$) mappings defined as follows:

$$T_n\left(\frac{k}{n}\right) = \begin{cases} 0, & k \in [0,n] \text{ is an even integer,} \\ 1, & k \in [0,n] \text{ is an odd integer,} \end{cases}$$

$$T_{n^-}\left(\frac{k}{n}\right) = \begin{cases} 1, & k \in [0,n] \text{ is an even integer,} \\ 0, & k \in [0,n] \text{ is an odd integer,} \end{cases}$$

and T_n, T_{n^-} are linear between these points.

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and T_n, T_{n^-} are linear between these points.

We say that a continuous function $f : [0,1] \to [0,1]$ has an *n-horseshoe* ($n \geq 2$) if there exist n closed subintervals of $[0,1]$, $A_1, \ldots, A_n$, with pairwise disjoint interiors, such that

$$(A_1 \cup \ldots \cup A_n) \subset (f(A_1) \cap \ldots \cap f(A_n)).$$

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$$(A_1 \cup \ldots \cup A_n) \subset (f(A_1) \cap \ldots \cap f(A_n)).$$

A continuous map $f : [0,1] \to [0,1]$ is called *turbulent* if it has a 2-horseshoe.

L. Block, J. Keesling, D. Ledis, *Semi-conjugacies and inverse limit spaces*, J. Difference Equ. Appl. 18 (2012), 627–645.

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Theorem 5

If $f : [0,1] \to [0,1]$ *is a turbulent map, then there exists a topological semi-conjugacy between* f *and* T_2*.*

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Theorem 5

If $f : [0,1] \to [0,1]$ is a turbulent map, then there exists a topological semi-conjugacy between f and T_2.

Example 1

Let $f : [0,1] \to [0,1]$ be given by

$$f(x) = \begin{cases} 2x + \frac{1}{2}, & x \in [0, \frac{1}{4}], \\ -2x + \frac{3}{2}, & x \in [\frac{1}{4}, \frac{3}{4}], \\ 2x - \frac{3}{2}, & x \in [\frac{3}{4}, 1]. \end{cases}$$

Then there exists a topological semi-conjugacy between f and T_2, but f is not turbulent.

Remark 2

The topological semi-conjugacy between a turbulent map and T_2 is not necessarily unique. For instance, there are two different topological semi-conjugacies between T_4 and T_2.

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Theorem 6

If $f : [0,1] \to [0,1]$ has an n-horseshoe, then there exists a topological semi-conjugacy between f and T_n or T_{n^-}.

D.-S. Ou, K.J. Palmer, *A constructive proof of the existence of a semi-conjugacy for a one dimensional map*, Discrete Contin. Dyn. Syst. Ser. B 17 (2012), 977–992.

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Theorem 7

If $f : [0,1] \to [0,1]$ is a weak horseshoe map having n laps and $f(0) = 0$, then there exists a unique increasing topological semi-conjugacy $\varphi : [0,1] \to [0,1]$ between f and T_n such that $\varphi(0) = 0$.

Denote by $\mathcal{M}_n(I)$ the set of all n-modal maps $f : I \to I$ and put

$$\mathcal{M}_n^1(I) := \{f \in \mathcal{M}_n(I) : f(x) < x \text{ for } x \in (t_0, t_1],$$

$$f(t_{2j}) = t_0 \text{ for } j \in \{0, \ldots, \lfloor\frac{n}{2}\rfloor\},$$

$$f(t_{2j-1}) = f(t_1) \geq f(t_{n+1}) \text{ for } j \in \{2, \ldots, \lfloor\frac{n+1}{2}\rfloor\}\},$$

where $(t_j)_{j=0}^{n+1}$ is the partition of the interval I.

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where $(t_j)_{j=0}^{n+1}$ is the partition of the interval I.

Y.-G. Shi, L. Li, Z. Leśniak, *On conjugacy of r-modal interval maps with non-monotonicity height equal to 1*, J. Difference Equ. Appl. 19 (2013), 573–584.

Theorem 8

Assume that $f \in \mathcal{M}_n^1(I)$ and $g \in \mathcal{M}_n^1(J)$. Then f and g are topologically conjugate if and only if there exists a positive integer m such that one of the following conditions holds:

(i) $f(t_{n+1}) = f^m(t_1)$, $g(s_{n+1}) = g^m(s_1)$,

(ii) $f(t_{n+1}) = t_0$, $g(s_{n+1}) = s_0$,

(iii) $f(t_{n+1}) \in \left(f^{m+1}(t_1), f^m(t_1)\right)$, $g(s_{n+1}) \in \left(g^{m+1}(s_1), g^m(s_1)\right)$.

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Assume that $f \in \mathcal{M}_n^1(I)$ *and* $g \in \mathcal{M}_n^1(J)$. *Then* f *and* g *are topologically conjugate if and only if there exists a positive integer* m *such that one of the following conditions holds:*

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(iii) $f(t_{n+1}) \in \left(f^{m+1}(t_1), f^m(t_1)\right)$, $g(s_{n+1}) \in \left(g^{m+1}(s_1), g^m(s_1)\right)$.

Moreover, in cases (i) and (ii) any homeomorphism $\varphi_0 : [f(t_1), t_1] \to [g(s_1), s_1]$ *satisfying*

$$\varphi_0(t_1) = s_1, \quad \varphi_0(f(t_1)) = g(s_1) \tag{4}$$

can be uniquely extended to a topological conjugacy between f *and* g. *The same is also true in case (iii) provided additionally*

$$\varphi_0(f^{-m}_{|[t_0,t_1]}(f(t_{n+1}))) = g^{-m}_{|[s_0,s_1]}(g(s_{n+1})). \tag{5}$$

Theorem 9

Under the assumptions of Theorem 8, any homeomorphism $\varphi_0 : [f(t_1), t_1] \to [g(s_1), s_1]$ *satisfying (4) (and (5) in case (iii)) can be extended to finitely many continuous non-monotone solutions of equation (1).*

Theorem 9

Under the assumptions of Theorem 8, any homeomorphism $\varphi_0 : [f(t_1), t_1] \to [g(s_1), s_1]$ *satisfying (4) (and (5) in case (iii)) can be extended to finitely many continuous non-monotone solutions of equation (1). If* $I = J$*, then only one of these solutions belongs to* $\mathcal{M}_n^1(I)$*.*