Totally internally incomparable families of permutations

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Object of our interests is a phenomenon of existence of the infinite families of permutations on $\mathbb{N}$ which are totally internally incomparable. This concept concerns only the divergent permutations – permutation p on $\mathbb{N}$ is the divergent permutation if there exists a conditionally convergent series $\sum a_n$ such that the $p-$rearranged series $\sum a_{p(n)}$ is divergent.

We discuss here only the real series.

Definition 1
We say that a nonempty family $\mathfrak{T}$ of divergent permutations is totally internally incomparable if for any subset $\mathfrak{S}$ of $\mathfrak{T}$, $\mathfrak{S} \neq \emptyset$ and $\mathfrak{S} \neq \mathfrak{T}$, we have

$$\bigcap_{s \in \mathfrak{S}} \sum(s) \setminus \bigcup_{t \in (\mathfrak{T} \setminus \mathfrak{S})} \sum(t) \neq \emptyset,$$

where $\sum(p)$ denotes the so called convergence class of permutation p, i.e. the family of all conditionally convergent series $\sum a_n$ such that the $p-$rearranged series $\sum a_{p(n)}$ is also convergent.

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Definition 2

A nonempty family $\mathfrak{T}$ of divergent permutations is internally incomparable if for any permutation $p \in \mathfrak{T}$ we have

$$\sum(p) \setminus \bigcup_{t \in \mathfrak{T} \setminus \{p\}} \sum(t) \neq \emptyset \quad and \quad \bigcap_{t \in \mathfrak{T} \setminus \{p\}} \sum(t) \setminus \sum(p) \neq \emptyset.$$

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The main object of our research is the question whether the following implication holds true:

internal incomparability $\Rightarrow$ *total internal incomparability – at least within the countable infinite families of permutations on* $\mathbb{N}$.

A problem of softer nature arises as well (within any families of permutations on $\mathbb{N}$), concerning the middle kinds of internal incomparability, more precisely: do they exist the families of permutations on $\mathbb{N}$ with internal incomparability stronger than the usual internal incomparability (**Definition 2**) and simultaneously weaker than the total internal incomparability (**Definition 1**)?

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Some results in this subject have been already investigated and discussed by us previously. Some selected results will be presented at the end of this elaboration.

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Theorem 1 (simple version)
Let $\mathfrak{T}$ be a finite family of divergent permutations. If $\mathfrak{T}$ is internally incomparable then $\mathfrak{T}$ is even totally internally incomparable.

Remark 1

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The respective example is very exciting, however it will not be presented here.

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Remark 2

The relation of internal incomparability of the countable family $\mathfrak{T}$ implies the condition

$$\bigcap_{t \in \mathfrak{T} \setminus \{p\}} \sum(t) \setminus \sum(p) \neq \emptyset,$$

which could be essentially strengthened to the following theorem.

Theorem 2 (another extension of Riemann's Derangement Theorem)
Let $\{\rho_k\}_{k=0}^{\infty}$ *be the sequence of divergent permutations and let* $\sum a_n$ *be a conditionally convergent series such that* $\sum a_{\rho_0(n)}$ *is divergent and all the series* $\sum\limits_{n=1}^{\infty} a_{\rho_k(n)}$, $k \in \mathbb{N}$, *are convergent.*
Then for every closed interval I *of* $[-\infty, \infty]$ *there exists a conditionally convergent series* $\sum b_n$ *such that interval* I *is the set of limit points of the series* $\sum b_{\rho_0(n)}$ *and all the series* $\sum b_{\rho_k(n)}$, $k \in \mathbb{N}$, *are convergent.*

Remark 3

Our friends, mathematicians as well, asked us ironically whether we know anybody (different from us), who in the period of last few years committed a paper devoted to the subject matter of Riemann's Derangement Theorem (for shortness denoted as "R.D.T.").

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*We have a pleasure to inform that absolutely yes. One of the freshest papers has been written by Jürgen Grahl and Shahar Nevo (famous for their results in the field of complex analysis) and is entitled: "On Riemann's Theorem about conditionally convergent series", New Zealand J. Math. 43 (***2013***), 85–93, where R.D.T. is extended to "multiple instead of simple series".*

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One of the main applications of their main result is the construction of an example of a continuous function $f : [0, \infty)^n \to \mathbb{R}$, *the iterated integrals of which exist for the each order of integration, but all of them have different values!*

Now we present one of our main results concerning the extension of countable families of permutations possessing the internal incomparability.

Theorem 3

Let $\mathfrak{T}$ be a countable family of permutations either internally incomparable or totally internally incomparable. Then there exist the two-sided divergent permutation p and the one-sided divergent permutation q such that both sets of permutations $\{p\} \cup \mathfrak{T}$ and $\{q\} \cup \mathfrak{T}$ are also internally incomparable or totally internally incomparable, respectively.

Let us here recall that permutation p on $\mathbb{N}$ is the two-sided divergent permutation if p and p^{-1} are both the divergent permutations.

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Whereas p is the one-sided divergent permutation on $\mathbb{N}$ if only p is the divergent permutation (i.e. p^{-1} is the convergent permutation in our terminology).

Remark 4

From Theorem 3 it results that each countable set of divergent permutations, which is either internally incomparable or totally internally incomparable, cannot be included in any countable set, maximal with respect to the inclusion relation, which is either internally incomparable or totally internally incomparable as well, respectively.

Remark 4

From Theorem 3 it results that each countable set of divergent permutations, which is either internally incomparable or totally internally incomparable, cannot be included in any countable set, maximal with respect to the inclusion relation, which is either internally incomparable or totally internally incomparable as well, respectively.

Remark 5

We do not know whether (without assuming CH) there exist the totally internally incomparable sets $\mathfrak{T}$ of divergent permutations such that $\operatorname{card}\mathfrak{T} = \mathfrak{c}$.

Algebraic type of internal incomparability

In the course of discussing the issue of internal incomparability of families of divergent permutations we considered also, in our sense, the algebraic version of this subject, called by us as the $d-$property.

Definition 3

A family D of divergent permutations possesses the $d-$property ($d-$one-sided property, $d-$two-sided property and d_n-property, d_n-one-sided property, d_n-two-sided property, respectively) if composition $\rho_1 \circ \rho_2 \circ ... \circ \rho_k$ is a divergent permutation (one-sided divergent permutation or two-sided divergent permutation, respectively) for every finite one-to-one sequence $\rho_1, \rho_2, ..., \rho_k$ of elements from D (with $k \leq n$ for the cases with d_n symbol).

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We note that the family $D = \mathfrak{DC}$ of all one-sided divergent permutations possesses the $d-$one-sided property, but the family $D = \mathfrak{DD}$ of all two-sided divergent permutations does not possess even the $d-$property. The last fact follows easily from the result given below.

Theorem 4

For every $n \in \mathbb{N}$, $n \geq 2$, there exist the two-sided divergent permutations $\rho_1, \rho_2, ..., \rho_n$ pairwise disjoint such that

$$\rho_1 \circ \rho_2 \circ ... \circ \rho_n = \mathrm{id}_{\mathbb{N}}$$

and set $\{\rho_1, \rho_2, ..., \rho_n\}$ possesses the $d_{n-1}-$two-sided property.

From the Kuratowski-Zorn lemma it results that each set of divergent permutations possessing the $d-$property is included in some set, maximal with respect to the inclusion relation, possessing the $d-$property (for shortness called as the max $d-$property set). It appeared that no max $d-$property set is a countable set. It follows easily from the following fact.

Theorem 5

If A is a countable set of divergent permutations then there exists a divergent permutation $p \notin A$ such that the semigroup generated by $A \cup \{p\}$ also contains only the divergent permutations.

Remark 6

Let $\mathfrak{D}$ and $\mathfrak{E}$ be the semigroups of permutations on $\mathbb{N}$. Then we have

$$Semigroup(\mathfrak{D} \cup \mathfrak{E}) =$$
$$\mathfrak{D} \cup \mathfrak{D}\mathfrak{E} \cup \mathfrak{D}\mathfrak{E}\mathfrak{D} \cup \mathfrak{D}\mathfrak{E}\mathfrak{D}\mathfrak{E} \cup \ldots \cup \mathfrak{E} \cup \mathfrak{E}\mathfrak{D} \cup \mathfrak{E}\mathfrak{D}\mathfrak{E} \cup \mathfrak{E}\mathfrak{D}\mathfrak{E}\mathfrak{D} \cup \ldots$$

Moreover, we have

$$Semigroup(\mathfrak{D} \cup \{p\}) = Semigroup(\mathfrak{D} \cup \mathfrak{E}),$$

where $\mathfrak{E} = \{p^k : \ k \in \mathbb{N}\}$ and $\mathbb{N}$ denotes the set of positive integers.

References and final comments

1^0 R.Wituła, D.Słota, *The convergence classes of divergent permutations*, Demonstratio Math. **42** (2009), 781-796.

In this paper, among other facts, we prove that for every divergent permutation p on $\mathbb{N}$ there exists a family $\Phi(p)$ of divergent permutations, with $\operatorname{card}\Phi(p) = \mathfrak{c}$, such that

$$\sum(p) \subset \sum(q)$$

for every $q \in \Phi(p)$ and $\{q_1, q_2\}$ is internally incomparable for any two $q_1, q_2 \in \Phi(p)$.

2^0 R.Wituła, *Diminution of the convergence classes of divergent permutations*, Studia Sc. Math. Hungarica Math. **50** (2013), 242-257.

It is proved in this paper that for each permutation p on $\mathbb{N}$ there exists a family Φ of one-sided divergent permutations with the power of the continuum such that
(i) for each $\varphi \in \Phi$ we have

$$\sum(p\varphi) \subset \sum(p)$$

and even

$$\sum(p) \setminus \bigcup_{\varphi\in\Phi} \sum(p\varphi) \neq \emptyset,$$

(ii) each of two families Φ and $p \circ \Phi$ is finitely internally incomparable.

3^0 R.Wituła, M.J. Przybyła, *The strongly and weakly divergent permutations*, Demonstratio Math. **39** (2006), 107-116.

Thank you
for your attention