Real functions in generalized probability II

Roman Frič [1,2] Martin Papčo, [2,1]

[1] Mathematical Institute, Slovak Academy of Sciences

[2] Catholic University in Ružomberok

This work was supported by
the Slovak Research and Development Agency [contract No. APVV-0178-11];
and Slovak Scientific Grant Agency [VEGA project 2/0046/11]

Abstract

OUR GOAL: To discuss two categorical constructions for bold algebras and mention some applications.

WHY bold algebras?

- MV-algebra = useful generalization of the Boolean algebra with applications to manyvalued logics;
- Bold algebra = MV-algebra of [0,1]-valued functions, equipped with Łukasziewicz operations;
- Bold algebras play important role in fuzzy probability theory.

Details:

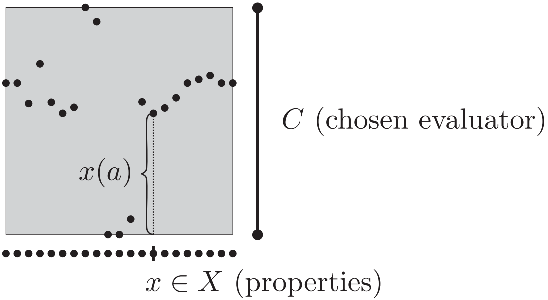
R. Frič, M. Papčo: On probability domains III. (Manuscript.)

A categorical approach to domains of probability

- Start with a "system $\mathcal{A}$ of events";
- Choose an "evaluator C"—a cogenerator; usually a structured set suitable for "evaluating" (e.g. the two element Boolean algebra, unit interval carrying Łukasiewicz MV-structure,. . .);
- Choose a set X of "properties"(separating $\mathcal{A}$);
- Represent each event $a \in \mathcal{A}$ via its evaluation $a_X \equiv \{x(a);\ x \in X\} \in C^X$, $a(x)$ denotes "how big" a is at x, $\mathcal{A}_X = \{a_X;\ a \in \mathcal{A}\}$;
- Form the minimal "subalgebra" D of C^X containing $\mathcal{A}_X$;
- The subalgebra forms a probability domain $D \subseteq C^X$;
- Observables are morphisms, states are observables into C.

Representation of an event

piece of reality	reality captured by evaluator
a (event) $\longmapsto$	a_X (evaluation of a)



Typical domains of probability

EXAMPLES

1. $\mathbb{A} \subseteq \{0,1\}^X$ ... a σ-field of subsets of X, classical probability theory (CPT)
2. $\mathcal{M}(\mathbb{A}) \subseteq [0,1]^X$ ... the measurable fuzzy subsets of X, fuzzy probability theory (FPT)

A categorical approach enables

- to see FPT as a fuzzification of CPT
- to study various generalized probability theories (GPT) using the same methodology based on D-posets (equivalently, on effect algebras)
- to consider generalized observables and generalized states as morphisms

Classical Probability Theory – Equipment

1. *Probability space* $(\Omega, \mathbf{A}, P)$,
   - Ω ... the set of all outcomes of a random experiment;
   - $\mathbf{A}$ ... σ-field of subsets of Ω; each $A \in \mathbf{A}$ is called an *event*;
   - each A is representable by its indicator function $\chi_A \colon \Omega \longrightarrow \{0, 1\}$, $\chi_A(\omega) = 1$ for $\omega \in A$ and $\chi_A(\omega) = 0$ otherwise
   - $P \colon \mathbf{A} \to [0, 1]$ ... a normed σ-additive measure called *probability*;
     $P(A)$ measures how "big" is $A \in \mathbf{A}$ in comparison to Ω
2. The most important example:
   $(R, \mathbf{B}_R, p)$, where R are the real numbers, $\mathbf{B}_R$ is the σ-field of Borel subsets of R, and p is a probability on $\mathbf{B}_R$
3. Let f be a measurable map of Ω into R, i.e.,
   $f^{\leftarrow}(B) = \{\omega \in \Omega \mid f(\omega) \in B\} \in \mathbf{A}$ for all $B \in \mathbf{B}$.
   If $p(B) = P(f^{\leftarrow}(B))$ for all $B \in \mathbf{B}$, then:
   f ... a *random variable*, p ... the *distribution of* f.

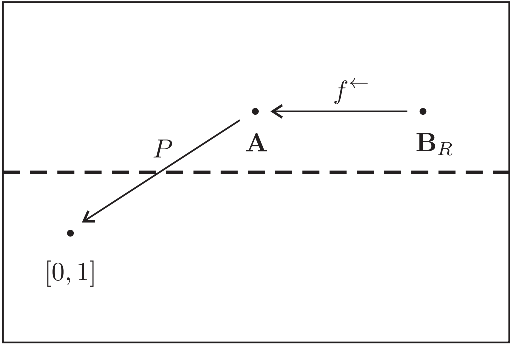
Conceptual disadvantages of CPT

- **A** is a Boolean algebra and it cannot be applied to situations where multivalued logic and fuzzy mathematics is needed.
- P as an additive map does not preserve the Boolean operations with events.
- The range of P is $[0, 1]$ and the domain of the dual map $f^{\leftarrow}$ (important in GPT) is **A**.

$\Rightarrow$ The two important maps in CPT are of a rather different nature and as such do not fit the categorical approach to probability

: both **A** and $[0, 1]$ should be objects, all relevant maps should be morphisms (preserving the structure of objects) and, if g and h are morphisms such that the range of g and the domain of h coincide, then the composition $h \circ g$ is a morphism, too.

Two maps



$B \in \mathbf{B}_R \subseteq \{0,1\}^R$

$A \in \mathbf{A} \subseteq \{0,1\}^\Omega$

$P(A) \in [0,1]$

HOW to fix the problem

Usual trick in mathematics:
To embed basic notions of CPT into a broader context.

- Potential objects will be (generalized) random events (equipped with additional mathematical structures: partial order, partial algebraic operations; convergence);
- Morphisms will be "observables" (structure preserving maps from one object to another);
- Crucial: Both probability measures and maps $f^{\leftarrow}$ of $\mathbf{B}_R$ into $\mathbf{A}$ should be morphisms.

Category theory offer:
Various epireflections implicitly used in CPT will be useful.

WHAT is an epireflection

EPIREFLECTION = (roughly) an embedding of a given object $\mathcal{O}$ into another object $e(\mathcal{O})$, so that $\mathcal{O}$ and its epireflection $e(\mathcal{O})$ "have the same good properties", but $e(\mathcal{O})$ "has some more" (enabling advanced constructions).

Examples:

1. The embedding of the field of rational numbers Q into the real numbers R (a completion suitable for real analysis);
2. The embedding of a field of sets $\mathbf{A}_0$ into the generated σ-field $\mathbf{A} = \sigma(\mathbf{A}_0)$ (a completion suitable for limit stochastics).

1. $\mathbf{A}_0$ ... a field of subsets of X, $\sigma(\mathbf{A}_0)$ ... the generated σ-field, p ... a probability measure on $\mathbf{A}_0$

   OBSERVATION: $\mathbf{A}_0$ and $\sigma(\mathbf{A}_0)$ "have the same probability measures".

2. $\mathbf{T}$ ... the trivial σ-field $\{\emptyset, \{a\}\}$ of subsets of a singleton $\{a\}$

   OBSERVATION: $\mathbf{T}$ and $\{0,1\} = \{0,1\}^{\{a\}}$ can be identified and $[0,1]$ can be considered as the Łukasiewicz tribe $\mathcal{M}(\mathbf{T})$ of all $\mathbf{T}$-measurable functions into $[0,1]$.

   There are two cases.

   (i) The range of p is $\{0,1\}$. Then p can be considered as a sequentially continuous Boolean homomorphism of $\mathbf{A}_0$ into $\mathbf{T}$;
   (ii) $0 < p(A) < 1$ for some $A \in \mathbf{A}_0$. Then p fails to be a Boolean homomorphism into $\mathbf{T}$.

3. OBSERVATION: *p preserves a D-poset structure*
—partial order, partial binary difference operation, associativity, two constants (bottom and top elements), existence of a complement, and a sequential convergence.

CONSEQUENCE: *p is a sequentially continuous D-homomorphism* of $\mathbf{A}_0$ into $\mathcal{M}(\mathbf{T})$ and *p_σ is a sequentially continuous D-homomorphism* of $\sigma(\mathbf{A}_0)$ into $\mathcal{M}(\mathbf{T})$.

4. $\mathbf{A}_0$ ... a field of subsets of X,
$\mathbf{A} = \sigma(\mathbf{A}_0)$ ... the generated σ-field,
$\mathcal{M}(\mathbf{A})$ ... the set of all $\mathbf{A}$-measurable functions into $[0,1]$

It is known that

(i) For each probability measure p on $\mathbf{A}$, the probability integral $\int f \, \mathrm{d}p$, $f \in \mathcal{M}(\mathbf{A})$, defines a sequentially continuous D-homomorphism of $\mathcal{M}(\mathbf{A})$ into $\mathcal{M}(\mathbf{T})$;

(ii) For each sequentially continuous D-homomorphism h of $\mathcal{M}(\mathbf{A})$ into $\mathcal{M}(\mathbf{T})$ there is a unique probability measure p on $\mathbf{A}$ such that $h(f) = \int f \, \mathrm{d}p$, $f \in \mathcal{M}(\mathbf{A})$;

(iii) $\mathbf{B}$ ... a σ-field of subsets of Y,
$\mathcal{M}(\mathbf{B})$ ... the set of all $\mathbf{B}$-measurable functions into $[0,1]$,
h ... a sequentially continuous D-homomorphism of $\mathbf{A}_0$ into $\mathcal{M}(\mathbf{B})$.

Then there is a sequentially continuous D-homomorphism $\overline{h}$ of $\mathcal{M}(\mathbf{A})$ into $\mathcal{M}(\mathbf{B})$ extending h and the extension is uniquely determined.

Further, if $\mathbf{B}_0$ is a field of subsets of Y such that $\mathbf{B} = \sigma(\mathbf{B}_0)$ and $h(\mathbf{A}_0) \subseteq \mathbf{B}_0$, then $\overline{h}(\mathbf{A}) \subseteq \mathbf{B}$ and the restriction of $\overline{h}$ to $\mathbf{A}$ is a sequentially continuous D-homomorphism of $\mathbf{A}$ into $\mathbf{B}$ and, moreover, it is a Boolean homomorphism.

Epireflection

$$
\begin{tikzcd}
\mathbf{A}_0 \arrow[d, "e"'] \arrow[dr, "h"] & \\
\mathcal{M}\big(\sigma(\mathbf{A}_0)\big) \arrow[r, dashed, "\overline{h}"] \arrow[ur, phantom, "\#"] & \mathcal{M}(\mathbf{B})
\end{tikzcd}
$$

Bold algebras on stage, please!

- Fields of sets $\mathbf{A}_0$, $\mathbf{A} = \sigma(\mathbf{A}_0)$, $\mathbf{B}_R$, and $\mathbf{T} = \{0, 1\}$ can be considered as bold algebras of $\{0, 1\}$-valued functions (indicator functions);
- Spaces of functions $\mathcal{M}(\mathbf{A})$, $\mathcal{M}(\mathbf{B}_R)$, and $\mathcal{M}(\mathbf{T}) = [0, 1]$ can be considered as bold algebras of $[0, 1]$-valued functions (fuzzy sets);
- The Boolean homomorphism $f^{\leftarrow}$ of $\mathbf{B}_R$ into $\mathbf{A}$ can be considered as the restriction of a morphism of $\mathcal{M}(\mathbf{B}_R)$ into $\mathcal{M}(\mathbf{A})$;
- The probability integral $\int (.)\, \mathrm{d}p$ is a morphism of $\mathcal{M}(\mathbf{A})$ into $\mathcal{M}(\mathbf{T}) = [0, 1]$, and p is its restriction to $\mathbf{A}$.

CONSEQUENCE: The category BID of bold algebras as objects and sequentially continuous D-homomorphisms as morphisms is exactly what is needed.

Bold algebras in FPT's service

- Fuzzy random events are modeled by the bold algebras of the form $\mathcal{M}(\mathbf{A})$ – they are characterized as complete (sequentially closed) divisible bold algebras.
- They form an epireflective (full) subcategory CGBID of BID.
- Morphisms of $\mathcal{M}(\mathbf{B})$ into $\mathcal{M}(\mathbf{A})$ serve as fuzzy observables and morphisms of $\mathcal{M}(\mathbf{B})$ into $\mathcal{M}(\mathbf{T})$ (probability integrals) serve as fuzzy probability measures.

BENEFIT: The "fuzzification" epireflector functor F sending an object $\mathcal{O}$ of BID into an object $F(\mathcal{O})$ of CGBID enables us to describe the transition from CPT to FPT.

FTP can be considered as a genuine categorical extension of CPT: the epireflector F "embeds" CPT into FPT.

NOTE: In FPT there are (quantum) phenomena which cannot be modeled within CPT.

(i) to each σ-field $\mathbf{A}$ of random events in CPT, there corresponds its fuzzified field of random events $\mathcal{M}(\mathbf{A})$
—the correspondence is one-to-one;

(ii) to each (generalized) observable $h\colon \mathbf{B} \longrightarrow \mathbf{A}$ there corresponds its fuzzified observable $F(h)\colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ (the correspondence between classical and fuzzy observables fails to be one-to-one);

(iii) each probability measure $p\colon \mathbf{A} \longrightarrow \mathcal{M}(\mathbf{T}) = [0,1]$ is the restriction of the fuzzy observable $F(p)\colon \mathcal{M}(\mathbf{A}) \longrightarrow \mathcal{M}(\mathbf{T})$, $F(p) = \int (.)\, \mathrm{d}p$;

(iv) there is a fuzzy observable $h\colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$, such that no observable in CPT is its restriction, mapping $\mathbf{B}$ into $\mathbf{A}$; such "genuine" fuzzy observables have distinct quantum quality (a crisp event is mapped to a genuine fuzzy event, e.g., for $A \in \mathbf{A}$ its probability $p(A)$, $0 < p(A) < 1$, can be considered as a fuzzy event).

$$
\begin{tikzcd}
\mathbf{T}=\{0,1\} & \mathbf{A} \arrow[dl, "p"'] & \mathbf{B} \arrow[l, "h"'] & \mathrm{BID} \arrow[d, "F"] \\
\mathcal{M}(\mathbf{T})=[0,1] & \mathcal{M}(\mathbf{A}) \arrow[l, "F(p)"'] & \mathcal{M}(\mathbf{B}) \arrow[l, "F(h)"'] & \mathrm{CGBID}
\end{tikzcd}
$$

B. Coecke: a category is the exact mathematical structure of practising physics!

Remember, FPT has the following QUANTUM PROPERTY:
a fuzzy observable can send a crisp event to a genuine fuzzy one and, dually, a fuzzy random variable can send a point measure (a classical elementary event) to a genuine probability measure

Further improvement of FPT is motivated by

B. Coecke: Introducing categories to the practicing physicist. In: What is Category Theory (ed. G. Sica), Polimetrica, Monza, 2006, 45–74.

Modification of a D-poset notion in order to get a “tensor quality”:

- by forming the terminal object – the top and bottom elements can collapse into one
- by replacing the category of D-posets of fuzzy sets by a suitable monoidal (tensor) category

B. Coecke: to be able to conceive two systems as one whole and to consider the compound operations inherited from the operations on the individual systems, we pass from ordinary categories to a particular case of the 2-dimensional variant of categories

Definition (roughly)

a monoidal category (or tensor category) = a category C equipped with

- a bifunctor $\otimes : C \times C \longrightarrow C$ which is associative
- an object I which is both a left and right identity for $\otimes$

which meet certain coherence conditions to ensure that all the relevant diagrams commute
(a bifunctor, also known as a binary functor, is a functor in two arguments)

BIDT – rough description

- First, we also consider $[0,1]^X$ for $X = \emptyset$. It is the "degenerated" bold algebra containing the only element $[0,1]^\emptyset = \{\emptyset\}$ in which the top and the bottom elements coincide. It serves as the terminal object $\top$: each morphism into $\top$ is uniquely determined(!);
- Objects of BIDT are pairs $(\mathcal{X}, \mathcal{Y})$, where $\mathcal{X} \subseteq [0,1]^X$ and $\mathcal{Y} \subseteq [0,1]^Y$ are bold algebras, or $[0,1]^\emptyset$ ($\top$ is not excluded) and $(\mathcal{X}, \mathcal{Y})$ is their categorical product (the difference structure is defined coordinatewise);
- If $(\mathcal{X}, \mathcal{Y})$, $(\mathcal{U}, \mathcal{V})$ are two pairs and $f : \mathcal{X} \longrightarrow \mathcal{U}$, $g : \mathcal{Y} \longrightarrow \mathcal{V}$ are morphisms, then the pair (f, g) is the morphism in BIDT and the composition $(f_2, g_2) \circ (f_1, g_1) = (f_2 \circ f_1, g_2 \circ g_1)$ is defined as the pair of pararell compositions.

More information about good properties of a monoidal category (hence of BIDT) can be found in the cited paper by B. Coecke.

GPT based on BIDT

From the formal point of view:

- the objects $(\mathcal{X}, \mathcal{Y})$ of BIDT are special D-posets of fuzzy sets
- the morphisms of BIDT are special (sequentially continuous) D-homomorphisms

HENCE the results obtained in OUR PREVIOUS STUDIES (in terms of properties of random events: lattice, closedness with respect to limits, divisibility):

- a classification of D-posets of fuzzy sets;
- the duality of observables and random variables;
- the transition from crisp to genuine fuzzy events;
- probability measures as fuzzy observables;

CAN BE ADAPTED FOR THE GPT BASED ON BIDT.

OBSERVATION: a bold algebra $\mathcal{X} \subseteq [0,1]^X$ and the pairs $(\top, \mathcal{X})$, $(\mathcal{X}, \top)$ are isomorphic, hence FPT can be embedded into a GPT based on the category BIDT.

Conclusion

FPT is a fuzzification of CPT in which

- observables and states are morphisms;
- classical probability measures can be viewed as observables into $[0,1]$;
- a fuzzy random variable can map a classical elementary event into a nondegenerated probability measure and, dually, a fuzzy observable can map a crisp event to a genuine fuzzy event.

Further

- BIDT is a "tensor modification" of FPT;
- IF-probability can be studied within FPT;
- GPT - the objects of which are products (pairs) of bold algebras can be studied within FPT.

Problem

PROBLEM. Is BID an epireflective subcategory of ID?