Preservation of topological properties by maps

Slávka Vernárska

Matej Bel University, Banská Bystrica, Slovak Republic

28th International Summer Conference on Real Functions Theory
Stará Lesná

Contents

Let X and Y be topological spaces and $f : X \to Y$ be a map. (We do not assume that f is continuous.)

Let P be *a topological property* (property of a set which is invariant under homeomorphisms) like residual set, dense set, nowhere dense set,...

Under what conditions on f the following implications

- $A \subset X$ has property $P \Rightarrow f(A) \subset Y$ has property P,
- $B \subset Y$ has property $P \Rightarrow f^{-1}(B) \subset X$ has property P,

and the converse implications hold?

Noll, D., *On the Preservation of Baire Category under Preimages* :
Let $f : X \to Y$ be a continuous dense and nearly feebly open mapping acting from a Čech complete space X into completely regular Y. Then residual sets are carried over under f.

Topological space X is called *Čech complete* if

- X is completely regular and
- X possesses a *complete sequence*, i.e., a sequence of an open coverings $(\mathcal{U}_n)$ such that every filter $\mathcal{F}$ on X satisfying $\mathcal{F} \cap \mathcal{U}_n \neq \emptyset$ for all n has a cluster point

$$\left(\bigcap_{A \in \mathcal{F}} \overline{A} \neq \emptyset \right)$$

Theorem

Let X be a Čech complete space and let Y be a metric space. Let $f : X \to Y$ be a quasi-continuous and nearly feebly open function such that $\overline{f(X)} = Y$. Then f maps residual subsets of X onto residual subsets of Y.

The condition f being quasi-continuous cannot be omitted. (example)

	$A \Rightarrow f(A)$	$f^{-1}(B) \Rightarrow B$	$B \Rightarrow f^{-1}(B)$	$f(A) \Rightarrow A$
dense				
nowhere dense				

The following properties are equivalent:

1. for any dense set $A \subset X$, then $f(A)$ is so in Y,
2. any $B \subset Y$ is dense whenever $f^{-1}(B)$ is so in X,
3. $f(X)$ is dense in Y and f is somewhat continuous.

A map $f : X \to Y$ is said to be *somewhat continuous* if

$$\operatorname{Int} f^{-1}(V) \neq \emptyset$$

whenever V is an open subset of Y such that $f^{-1}(V) \neq \emptyset$.

	$A \Rightarrow f(A)$	$f^{-1}(B) \Rightarrow B$	$B \Rightarrow f^{-1}(B)$	$f(A) \Rightarrow A$
dense	$\overline{f(X)} = Y, \mathrm{SC}$			
	$\Leftrightarrow$			
nowhere dense				

The following properties are equivalent:

1. any $A \subset X$ is dense whenever $f(A)$ is so in Y,
2. f has no redundant open set.

A nonempty open set $U \subset X$ is said to be *redundant* for f, if $\overline{f(X - U)} = \overline{f(X)}$.

<table>
<tr><td></td><td>$A \Rightarrow f(A)$</td><td>$f^{-1}(B) \Rightarrow B$</td><td>$B \Rightarrow f^{-1}(B)$</td><td>$f(A) \Rightarrow A$</td></tr>
<tr><td>dense</td><td colspan="2">$\overline{f(X)} = Y$, SC</td><td></td><td>NROS</td></tr>
<tr><td></td><td colspan="2">$\Leftrightarrow$</td><td></td><td></td></tr>
<tr><td>nowhere dense</td><td></td><td></td><td></td><td></td></tr>
</table>

The following properties are equivalent:

1. for any dense set $B \subset Y$, $f^{-1}(B)$ is so in X,
2. f is feebly open.

A map $f : X \to Y$ is said to be *feebly open* if

$$\operatorname{Int} f(U) \neq \emptyset$$

whenever $U \neq \emptyset$ is an open subset of X.

<table>
<tr><th></th><th>$A \Rightarrow f(A)$</th><th>$f^{-1}(B) \Rightarrow B$</th><th>$B \Rightarrow f^{-1}(B)$</th><th>$f(A) \Rightarrow A$</th></tr>
<tr><td>dense</td><td colspan="2">$\overline{f(X)} = Y$, SC</td><td>FO</td><td>NROS</td></tr>
<tr><td></td><td colspan="2">$\Leftrightarrow$</td><td></td><td></td></tr>
<tr><td>nowhere dense</td><td></td><td></td><td></td><td></td></tr>
</table>

Consider the following two conditions.

1. $\operatorname{Int} f(X)$ is dense in Y and any $A \subset X$ is dense whenever $f(A)$ is so in Y,
2. for any dense set $B \subset Y$, $f^{-1}(B)$ is so in X.

Then (1) $\Rightarrow$ (2).

<table>
<tr><td></td><td>$A \Rightarrow f(A)$</td><td>$f^{-1}(B) \Rightarrow B$</td><td>$B \Rightarrow f^{-1}(B)$</td><td>$f(A) \Rightarrow A$</td></tr>
<tr><td>dense</td><td colspan="2">$\overline{f(X)} = Y, \mathrm{SC}$</td><td>FO</td><td>NROS</td></tr>
<tr><td></td><td colspan="2">$\Leftrightarrow$</td><td colspan="2">$\Leftarrow \overline{\mathrm{Int}\, f(X)} = Y$</td></tr>
<tr><td>nowhere dense</td><td></td><td></td><td></td><td></td></tr>
</table>

The following properties are equivalent:

1. for any nowhere dense set $B \subset Y$, $f^{-1}(B)$ is so in X,
2. any $A \subset X$ is nowhere dense whenever $f(A)$ is so in Y,
3. f is nearly feebly open and restrictively quasi-continuous.

A map $f : X \to Y$ is said to be *restrictively quasi-continuous* if, for every closed nowhere dense set $C \subset Y$,

$$\operatorname{Int} \overline{f^{-1}(C)} \subset f^{-1}(C).$$

<table>
<tr><td></td><td>$A \Rightarrow f(A)$</td><td>$f^{-1}(B) \Rightarrow B$</td><td>$B \Rightarrow f^{-1}(B)$</td><td>$f(A) \Rightarrow A$</td></tr>
<tr><td>dense</td><td colspan="2">$\overline{f(X)} = Y$, SC</td><td>FO</td><td>NROS</td></tr>
<tr><td></td><td colspan="2">$\Leftrightarrow$</td><td colspan="2">$\Leftarrow \overline{\operatorname{Int} f(X)} = Y$</td></tr>
<tr><td>nowhere dense</td><td></td><td></td><td colspan="2">NFO, RQC</td></tr>
<tr><td></td><td></td><td></td><td colspan="2">$\Leftrightarrow$</td></tr>
</table>

Consider the following two conditions.

1. $f(X)$ is dense in Y, f is somewhat continuous and has no redundant open set,
2. for any nowhere dense set $A \subset X$, $f(A)$ is so in Y.

Then (1) $\Rightarrow$ (2).

Example ((2) $\Rightarrow$ (1) is not true).

$$f(x) = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}), \\ 2 - 2x & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

<table>
<tr><th></th><th>$A \Rightarrow f(A)$</th><th>$f^{-1}(B) \Rightarrow B$</th><th>$B \Rightarrow f^{-1}(B)$</th><th>$f(A) \Rightarrow A$</th></tr>
<tr><td>dense</td><td colspan="2">$\overline{f(X)} = Y, \mathrm{SC}$</td><td>FO</td><td>NROS</td></tr>
<tr><td></td><td colspan="2">$\Leftrightarrow$</td><td colspan="2">$\Leftarrow \overline{\operatorname{Int} f(X)} = Y$</td></tr>
<tr><td>nowhere dense</td><td>$\Leftarrow \overline{f(X)} = Y$,
SC, NROS</td><td></td><td colspan="2">NFO, RQC</td></tr>
<tr><td></td><td></td><td></td><td colspan="2">$\Leftrightarrow$</td></tr>
</table>

Consider the two following conditions.

1. f is feebly open and somewhat continuous,
2. any $B \subset Y$ is nowhere dense whenever $f^{-1}(B)$ is so in X.

Then (1) $\Rightarrow$ (2).

Example ((2) $\Rightarrow$ (1) is not true).

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ -x & \text{if } x \in \mathbb{R} - \mathbb{Q}. \end{cases}$$

	$A \Rightarrow f(A)$	$f^{-1}(B) \Rightarrow B$	$B \Rightarrow f^{-1}(B)$	$f(A) \Rightarrow A$
dense	$\overline{f(X)} = Y$, SC		FO	NROS
	$\Leftrightarrow$		$\Leftarrow \overline{\mathrm{Int}\, f(X)} = Y$	
nowhere dense	$\Leftarrow \overline{f(X)} = Y$, SC, NROS	$\Leftarrow$ FO, SC	NFO, RQC	
			$\Leftrightarrow$	

Consider the two following conditions.

1. $\operatorname{Int} f(X)$ is dense in Y and for any nowhere dense set $A \subset X$, $f(A)$ is so in Y,
2. any $B \subset Y$ is nowhere dense whenever $f^{-1}(B)$ is so in X.

Then (1) $\Rightarrow$ (2).

<table>
<tr><td></td><td>$A \Rightarrow f(A)$</td><td>$f^{-1}(B) \Rightarrow B$</td><td>$B \Rightarrow f^{-1}(B)$</td><td>$f(A) \Rightarrow A$</td></tr>
<tr><td>dense</td><td colspan="2">$\overline{f(X)} = Y, \mathrm{SC}$</td><td>FO</td><td>NROS</td></tr>
<tr><td></td><td colspan="2">$\Leftrightarrow$</td><td colspan="2">$\Leftarrow \overline{\operatorname{Int} f(X)} = Y$</td></tr>
<tr><td>nowhere dense</td><td>$\Leftarrow \overline{f(X)} = Y$,
SC, NROS</td><td>$\Leftarrow$
FO, SC</td><td colspan="2">NFO, RQC</td></tr>
<tr><td></td><td colspan="2">$\Leftarrow \overline{\operatorname{Int} f(X)} = Y$</td><td colspan="2">$\Leftrightarrow$</td></tr>
</table>

Engelking, R. *General topology*, Sigma Series in Pure Mathematics, 6. Heldermann Verlag, Berlin, 1989.

Frolík, Z., *Remarks concerning the invariance of Baire spaces under preimages*, Czechoslovak Math. J. **11 (86)** (1961) 381–385.

Gentry, K. R., Hoyle, III, H. B., *Somewhat continuous functions*, Czechoslovak Math. J. **21 (96)** (1971) 5–12.

Kolyada, S., Snoha, Ľ., Trofimchuk, S., *Noninvertible minimal maps*, Fund. Math. **168** (2001), no. 2, 141–163.

Kuratowski, K., *Topology*, Vol. I, Academic Press, New York, 1966.

Łojasiewicz, S., *Wstęp do teorii funkcji rzeczywistych, Wydanie drugie*, Biblioteka matematyczna, Państwowe Wydawn. Naukowe, Warszawa, 1976.

Maličký, P., *Category version of the Poincaré recurrence theorem*, Topology Appl. **154** (2007), no. 14, 2709–2713.

Maličký, P., *Backward orbits of transitive maps*, J. Difference Equ. Appl. **18** (2012), no. 7, 1193–1203.

Neubrunn, T., *Quasi-continuity*, Real Anal. Exchange. **14** (1988/89), no. 2, 259–306.

Noll, D., *On the Preservation of Baire Category under Preimages*, Proc. Amer. Math. Soc., **107** (1989), 847–854.