Faculty of Mathematics
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On $\mathcal{J}$-continuous functions

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Table of contents

Definition 1 (H. Lebesgue).

A point $x_0 \in \mathbb{R}$ *is a* **density point** *of a Lebesgue measurable set* A *if*

$$\lim_{h \to 0^+} \frac{\lambda(A \cap [x_0 - h, x_0 + h])}{2h} = 1.$$

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$$\lim_{n \to \infty} \frac{\lambda\left(A \cap \left[x_0 - \frac{1}{n}, x_0 + \frac{1}{n}\right]\right)}{\frac{2}{n}} = 1.$$

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Theorem 1 (Lebesgue, 1910).

$$\lambda\left(A \bigtriangleup \Phi_d(A)\right) = 0.$$

Theorem 2.

For any sets $A, B \in \mathcal{L}$ we have :

1. $\Phi_d(\emptyset) = \emptyset, \quad \Phi_d(\mathbb{R}) = \mathbb{R};$
2. $\lambda(A \bigtriangleup B) = 0 \Rightarrow \Phi_d(A) = \Phi_d(B);$
3. $\lambda(A \bigtriangleup \Phi_d(A)) = 0;$
4. $\Phi_d(A \cap B) = \Phi_d(A) \cap \Phi_d(B).$

Theorem 3 (O. Haupt, C. Pauc, 1952).

The family

$$\mathcal{T}_d = \left\{A \in \mathcal{L}\colon A \subset \Phi_d(A)\right\}$$

forms a topology on $\mathbb{R}$ *called the density topology.*
Moreover, we have $\mathcal{T}_{nat} \subset \mathcal{T}_d$.

$\mathcal{J} = \{J_n\}_{n\in\mathbb{N}}$ is a sequence of intervals tending to zero if

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Definition 2.

*We shall say that a point $x_0 \in \mathbb{R}$ is a $\mathcal{J}$–**density point** of a measurable set A, if*

$$\lim_{n\to\infty} \frac{\lambda(A \cap (J_n + x_0))}{|J_n|} = 1.$$

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3. $\Phi_{\mathcal{J}}(A \cap B) = \Phi_{\mathcal{J}}(A) \cap \Phi_{\mathcal{J}}(B).$

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$$\underset{J\to 0}{\exists}\ \underset{A\in\mathcal{L}}{\exists}\quad \lambda(A)>0\wedge\lambda(A\cap\Phi_{\mathcal{J}}(A))=0.$$

Theorem 5.

If $A\in\mathcal{L}$ *then* $\Phi_{\mathcal{J}}(A)\in\mathcal{L}$.

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If $A\in\mathcal{L}$ *then* $\Phi_{\mathcal{J}}(A)\in\mathcal{L}$.

Theorem 6.

For $A\in\mathcal{L}$ *we have*

$$\lambda(\Phi_{\mathcal{J}}(A)\setminus A)=0.$$

Theorem 7.

Let $\mathcal{J}$ be a sequence of intervals tending to zero. Then

$$\mathcal{T}_{\mathcal{J}} = \{A \in \mathcal{L} : A \subset \Phi_{\mathcal{J}}(A)\}.$$

is a topology on $\mathbb{R}$, which will be called $\mathcal{J}$-density topology. Moreover, we have $\mathcal{T}_{nat} \subset \mathcal{T}_{\mathcal{J}}$.

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(iii) $(\mathbb{R}, \mathcal{T}_{\mathcal{J}})$ *is neither separable nor has the Lindelöf property;*

(iv) *a set A is compact with respect to $\mathcal{T}_{\mathcal{J}}$ if, and only if, it is finite.*

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\mathcal{C}_{nat,nat} =& \{f\colon (\mathbb{R}, \mathcal{T}_{nat}) \to (\mathbb{R}, \mathcal{T}_{nat})\},\\
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For $\mathcal{J} \in \Im$ the family $\mathcal{C}_{nat,\mathcal{J}}$ consists of constant functions.

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$f([a, b])$ is $\mathcal{T}_{\mathcal{J}}$-compact and $\mathcal{T}_{\mathcal{J}}$-connected.
$f([a, b])$ is a singleton, $f(a) = f(b)$ and f is constant.

Property 2.

For $\mathcal{J} \in \Im$ the following inclusions holds:

$$\begin{array}{lcl} \mathcal{C}_{nat,\mathcal{J}} \subsetneq & \mathcal{C}_{nat,nat} & \subset \mathcal{C}_{\mathcal{J},nat} \\ \mathcal{C}_{nat,\mathcal{J}} \subsetneq & \mathcal{C}_{\mathcal{J},\mathcal{J}} & \subset \mathcal{C}_{\mathcal{J},nat} \end{array}$$

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$f(x) = x$ is the member of $\mathcal{C}_{nat,nat}$ and $\mathcal{C}_{\mathcal{J},\mathcal{J}}$ but not $\mathcal{C}_{nat,\mathcal{J}}$.

We say that a sequence of intervals $\mathcal{J} = \{[a_n, b_n]\}_{n\in\mathbb{N}} \in \Im$, is **right-side (left-side) tending to zero** if there exists $n_0 \in \mathbb{N}$ such that $b_n > 0$ $(a_n < 0)$ for $n \geq n_0$ and

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Sequence of intervals $\mathcal{J} \in \Im$ is **one-side tending to zero** if it is right-side or left-side tending to zero.

Theorem 9.

Let $\mathcal{J} \in \Im$. Then $[a, b) \in \mathcal{T}_{\mathcal{J}}$ $((a, b] \in \mathcal{T}_{\mathcal{J}})$ for $a < b$ if and only if the sequence $\mathcal{J}$ is right-side (left-side) tending to zero.

Theorem 10.

If $\mathcal{J}$ is a sequence of intervals one-side tending to zero, then:

(i) $\mathcal{C}_{nat,nat} \setminus \mathcal{C}_{\mathcal{J},\mathcal{J}} \neq \emptyset$,

(ii) $\mathcal{C}_{\mathcal{J},\mathcal{J}} \setminus \mathcal{C}_{nat,nat} \neq \emptyset$.

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Theorem 11.

Let $\mathcal{J}$ be a sequence of intervals one-side tending to zero. Then

(i) $\mathcal{C}_{\mathcal{J},\mathcal{J}} \subsetneq \mathcal{C}_{\mathcal{J},nat}$,

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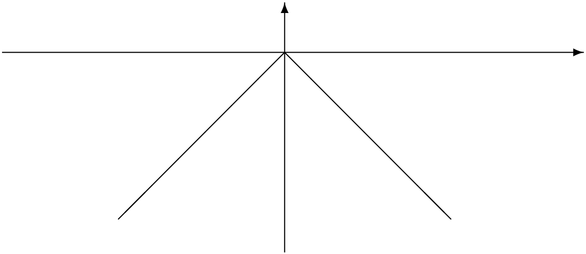
Let $\mathcal{J}$ be the sequence right side-tending to zero.
$f(x) = -|x|$, $f \in \mathcal{C}_{nat,nat} \subset \mathcal{C}_{\mathcal{J},nat}$.



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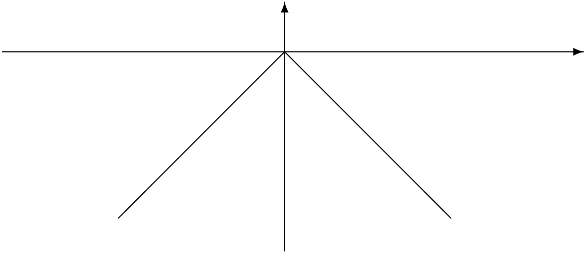

$[-1,1) \in \mathcal{T}_{\mathcal{J}}$ but $f^{-1}([-1,1)) = [-1,1] \notin \mathcal{T}_{\mathcal{J}}$. Thus $f \notin \mathcal{C}_{\mathcal{J},\mathcal{J}}$.

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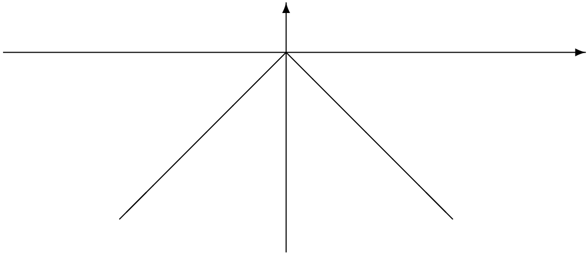

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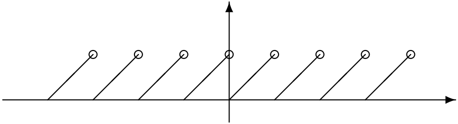
Let $\mathcal{J}$ be the sequence right side-tending to zero.

$$h(x) = x - k \qquad \text{for } x \in [k, k+1),\ k \in \mathbb{Z}.$$



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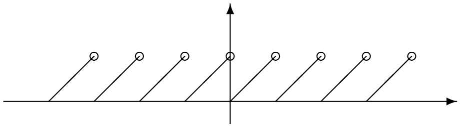

If $A \in \mathcal{T}_{\mathcal{J}}$, then $h^{-1}(A) = \bigcup_{k\in\mathbb{Z}} \big((A \cap [0,1)) + k\big) \in \mathcal{T}_{\mathcal{J}}$.
Hence $h \in \mathcal{C}_{\mathcal{J},\mathcal{J}} \subset \mathcal{C}_{\mathcal{J},nat}$.

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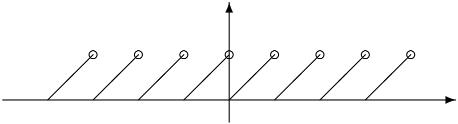

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Since $h^{-1}\left(\left(-1, \frac{1}{2}\right)\right) = \bigcup_{k\in\mathbb{Z}} \left[k, k+\frac{1}{2}\right) \notin \mathcal{T}_{nat}$, we have $h \notin \mathcal{C}_{nat,nat}$.

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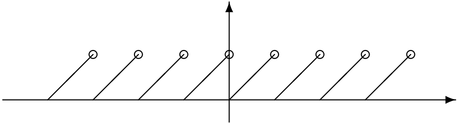

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$h \in \mathcal{C}_{\mathcal{J},\mathcal{J}} \setminus \mathcal{C}_{nat,nat}$, $h \in \mathcal{C}_{\mathcal{J},nat} \setminus \mathcal{C}_{nat,nat}$.

Property 3.

For $\mathcal{J} \in \Im$ be a sequence of intervals one-side tending to zero. Then the following inclusions holds:

$$\begin{array}{ccccc} \mathcal{C}_{nat,\mathcal{J}} & \subsetneq & \mathcal{C}_{nat,nat} & \subsetneq & \mathcal{C}_{\mathcal{J},nat} \\ \mathcal{C}_{nat,\mathcal{J}} & \subsetneq & \mathcal{C}_{\mathcal{J},\mathcal{J}} & \subsetneq & \mathcal{C}_{\mathcal{J},nat} \end{array}$$

Theorem 12.

If $\mathcal{J} \in \Im$, then there exists $\mathcal{K} \in \Im$ such that $\mathcal{T}_{\mathcal{J}} \neq \mathcal{T}_{\mathcal{K}}$, $\mathcal{C}_{\mathcal{J},\mathcal{J}} \neq \mathcal{C}_{\mathcal{K},\mathcal{K}}$ and $\mathcal{C}_{\mathcal{J},nat} \neq \mathcal{C}_{\mathcal{K},nat}$.

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$$\mathcal{T}_{\mathcal{J}} \neq \mathcal{T}_{\mathcal{K}} \iff \mathcal{C}_{\mathcal{J},nat} \neq \mathcal{C}_{\mathcal{K},nat}.$$

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Let $\mathcal{J}$, $\mathcal{K} \in \Im$. Then the sequence ordered in an arbitrary fashion containing all intervals of the sequences $\mathcal{J}$ and $\mathcal{K}$, denoted by $\mathcal{J} \cup \mathcal{K}$, is called **the union of sequences** $\mathcal{J}$ and $\mathcal{K}$.

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If $\mathcal{J} \in \Im$ *and* $\mathcal{K} \in \Im$, *then*

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It implies that $f \notin \mathcal{C}_{\mathcal{K},\mathcal{K}}$.

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