Density of the topology of uniform convergence on $C(X)$

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Outline

Definitions

- Let X be a Tychonoff space and Y be a metric space;
- let $C(X, Y)$ be a space of continuous functions from X to Y;
- let $C^*(X, Y)$ be a space of bounded continuous functions from X to Y;
- on these spaces we consider τ_U the topology of uniform convergence (generated by supremum metric);
- $d(X)$ denotes the density of X;
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Stone-Weierstrass Theorem

let $C(X) = C(X, R)$ and $C^*(X) = C^*(X, R)$;

we are interested in $d(C(X))$

Theorem (Stone-Weierstrass)

Suppose X is a compact Hausdorff space and A is a subalgebra of $C(X)$ which contains a non-zero constant function. Then A is dense in $C(X)$ iff it separates points.

Corollary

Let X be a compact Hausdorff space then $d(C(X)) = w(X)$.

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Čech-Stone Compactification

[R.A. McCoy and I. Ntantu, *Topological Properties of Spaces of Continuous Functions*. Springer-Verlag, 1988.]
Theorem 4.2.4 states this

Let X be a Tychonoff space then $w(\beta X) = d(C(X))$.

but from the proof we have only this

Theorem

Let X be a Tychonoff space then $w(\beta X) = d(C^*(X)) \leq d(C(X))$.

For which Tychonoff spaces X does $d(C(X)) = w(\beta X)$ hold?

Obviously for pseudocompact X.

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Compactness Degree

Definition

A topological space Z is called $\mathfrak{m}-$compact iff every open cover of Z has a subcover with the cardinality less than $\mathfrak{m}$. Put

$$\delta(Z) = \min\{\mathfrak{m};\ Z \text{ is } \mathfrak{m}\text{-compact}\}.$$

We will call $\delta(Z)$ the *compactness degree* of Z.

Proposition

For a topological space Z holds $L(Z) \leq \delta(Z) \leq L(Z)^{+}$.
(L is Lindelöff number.)

A metrizable space Z is *generalized compact (GK)* iff $\delta(Z) = L(Z)$

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Metrizable Spaces - via δ

Theorem (Theorem 2.6)

Let X be a metrizable space and Y be a path connected metric space. We have that

1. *if X is not GK and Y is not GTB, then $d(C(X,Y)) = d(Y)^{d(X)}$;*
2. *if X is not GK and Y is GTB, then $d(C(X,Y)) = [< d(Y)]^{d(X)}$;*
3. *if X is GK and Y is not GTB, then $d(C(X,Y)) = d(Y)^{<d(X)}$;*
4. *if X is GK and Y is GTB, then $d(C(X,Y)) = [< d(Y)]^{<d(X)}$.*

[C. Costantini. On the density of the space of continuous and uniformly continuous functions. *Topology and its Applications*, 153(7):1056–1078, 2006.]

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Corollary

Let X be a metrizable space and Y be a path connected metric space. We have that

1. *if Y is not GTB, then $d(C(X,Y)) = d(Y)^{<\delta(X)}$;*
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Corollary

Let X be a metrizable space and Y be a path connected separable metric space then $d(C(X,Y)) = 2^{<\delta(X)}$.

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Let X be a metrizable space then $d(C(X)) = w(\beta X) = 2^{<\delta(X)}$.

Corollary

Let X be a metrizable, Lindelöff and non-compact space, then $w(\beta X) = \mathfrak{c} = 2^{\omega}$

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Non-metrizable Spaces

Example

Let $X = [0, \omega_1)$ with the order topology. Since X is pseudocompact and $\beta X = [0, \omega_1]$ we have that $d(C(X)) = w(\beta X) = \omega_1$ and since $\delta(X) = \omega_2$ we have that $2^{<\delta(X)} = 2^{\omega_1} > \omega_1$.

- $C^*(X)$ is a connected component of 0 in $C(X)$;
- factor group (w.r. addition) $C(X)/C^*(X)$ represents the system of connected components of $C(X)$ which are all homeomorphic to $C^*(X)$;

Proposition

Let X be a Tychonoff space then $d(C(X)) = w(\beta X)|C(X)/C^*(X)|$.

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Countably Paracompact T_4 Spaces

Theorem

Let X be a countably paracompact T_4 space then $d(C(X)) \leq w(\beta(X \times I))$.

Proposition

Let X be a metrizable space or a pseudocompact space then $w(\beta(X \times I)) = w(\beta X)$.

For which Tychonoff spaces X does $w(\beta(X \times I)) = w(\beta X)$ hold?

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Summary

- Concerning the question when does $d(C(X)) = w(\beta X)$ hold; we know that for metrizable or pseudocompact X it is true;
- for other spaces it can help to study $C(X)/C^*(X)$;
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For Further Reading

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Thank You for Your Attention