On selections of set-valued maps satisfying some inclusions

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Theorem 1

Let X be a linear normed space, Y a Banach space and $\epsilon > 0$. Then for every function $f : X \to Y$ satisfying the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon, \qquad x, y \in X \tag{1}$$

there exists a unique additive function $g : X \to Y$ such that

$$\|f(x) - g(x)\| \leq \epsilon, \qquad x \in X. \tag{2}$$

D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941) 222–224.

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$$f(x+y) - f(x) - f(y) \in B(0, \epsilon), \qquad x, y \in X$$

$$f(x+y) + B(0,\epsilon) \subset f(x) + B(0,\epsilon) + f(y) + B(0,\epsilon), \qquad x, y \in X$$

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Theorem 2

Let $(S, +)$ be a commutative semigroup with zero, X a real Banach space and $F\colon S \to ccl(X)$ a set-valued map such that

$$F(x+y) \subset F(x) + F(y), \qquad x, y \in S$$

and $\sup\{\delta(F(x)) : x \in S\} < \infty$. Then F admits a unique additive selection.

Z. Gajda, R. Ger, *Subadditive multifunctions and Hyers-Ulam stability*, Numerical Mathematics, 80 (1987), 281–291.

Theorem 3

Let X be a real vector space, Y be a real Banach space, K be a convex cone in X, $a, b, p, q > 0$, $F\colon K \to ccl(Y)$,

$$F(ax + by) \subset pF(x) + qF(y) \qquad \text{for } x, y \in K$$

and $\sup\{\delta(F(x)) : \ x \in K\} < \infty$.

(i) *If $p + q > 1$, then there exists a unique selection $f : K \to Y$ of the multifunction F such that*

$$f(ax + by) = pf(x) + qf(y) \qquad \text{for } x, y \in K.$$

(ii) *If $p + q < 1$, then F is single-valued.*

D. Popa, *A stability result for a general linear inclusion*, Nonlinear Funct. Anal. App. 3 (2004), 405–414.

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K. Nikodem, D. Popa, *On selections of general linear inclusions*, Publ. Math. Debrecen 75 (2009), 239–249.

$$F(ax+by+k) \subset pF(x)+qF(y), \qquad x,y \in K,$$

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If

$$F(ax+by+k) \subset pF(x)+qF(y)+C, \qquad x,y \in K,$$

where C is a nonempty, compact and convex subset of Y,
$a+b \neq 1$, $p+q > 1$,

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$$f(ax+by+k) = pf(x)+qf(y), \qquad x,y \in K$$

and

$$f(x) \in F(x) + \frac{1}{p+q-1}C, \qquad x \in K.$$

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$G(x) = F(x) + \frac{1}{p+q-1}C$

Theorem 5

Assume that K *is a nonempty set,* (Y, d) *is a metric space. Let* $F\colon K \to n(Y)$, $\Psi\colon Y \to Y$, $\alpha\colon K \to K$, $\lambda \in (0, +\infty)$, $d(\Psi(x), \Psi(y)) \leq \lambda d(x, y)$ *for* $x, y \in Y$ *and* $\lim_{n\to\infty} \lambda^n \delta(F(\alpha^n(x))) = 0$ *for* $x \in K$.

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(i) *If Y is complete and*

$$\Psi(F(\alpha(x))) \subset F(x), \qquad x \in K,$$

then, for each $x \in K$, $\lim_{n\to\infty} \mathrm{cl}\,\Psi^n \circ F \circ \alpha^n(x) = f(x)$ exists and f is a unique selection of the multifunction $\mathrm{cl}\,F$ such that $\Psi \circ f \circ \alpha = f$.

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(ii) *If*

$$F(x) \subset \Psi(F(\alpha(x))), \qquad x \in K,$$

then F is a single-valued function and $\Psi \circ F \circ \alpha = F$.

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$$\Psi(x) = \frac{1}{p+q}x, \qquad \alpha(x) = (a+b)x$$

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$$\Psi(x)=\frac{1}{p+q}x, \qquad \alpha(x)=(a+b)x$$

$$\lim_{n\to\infty}\Psi^n(F(\alpha^n(x)))=\lim_{n\to\infty}\frac{1}{(p+q)^n}F((a+b)^n x)=f(x)$$

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$$\frac{F((a+b)^n(ax+by))}{(p+q)^n} \subset p\frac{F((a+b)^n x)}{(p+q)^n} + q\frac{F((a+b)^n y)}{(p+q)^n}, \qquad x,y \in K,$$

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Now, replacing x by $\frac{1}{a+b}x$ in the last inclusion we obtain

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F is single-valued and satisfies the equality

$$F(ax+by) = pF(x) + qF(y), \qquad x, y \in K.$$

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$F\colon \mathbb{R} \to ccl(\mathbb{R})$ given by $F(x) = [x-1, x+1]$ satisfies

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and each function $f(x)=x+b$, where $b\in[-1,1]$ is a Jensen selection of F.

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and each function $f(x) = x + b$, where $b \in [-1, 1]$ is a Jensen selection of F.
$F(x) = M$, where $M \in ccl(X)$ satisfies

$$F(ax + by) = pF(x) + qF(y)$$

and each constant function $f(x) = m$, where $m \in M$ satisfies $f(ax + by) = pf(x) + qf(y)$.

Let $(T, \star)$ be a groupoid, where $\star$ is square symmetric, i.e, $(x \star y) \star (x \star y) = (x \star x) \star (y \star y)$ for $x, y \in T$. Then the map $\rho\colon T \to T$ given by $\rho(x) := x \star x$ for $x \in T$ is an endomorfism of the groupoid $(T, \star)$.

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$$x \star y = ax + by + k, \qquad a, b > 0,\ x, y, k \in K$$

$$x \star y = \alpha(x) + \beta(y) + k, \qquad x, y \in T,\ k \in K$$

$\alpha, \beta \colon T \to K$ homomorphism and $\alpha \circ \beta = \beta \circ \alpha$

Corollary 1

Let $S \subset T$, $\rho(S) \subset S$, $a, b > 0$, Y be a real Banach space, $F\colon S \to ccl(Y)$, $\sup\{\delta(F(x)) :\ x \in S\} < \infty$ and

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(i) *If $p + q > 1$, then there exists a unique selection $f : S \to Y$ of the multifunction F such that*

$$f(x \star y) = pf(x) + qf(y) \qquad \text{for } x, y \in S,\ x \star y \in S.$$

(ii) *If $p + q < 1$ and ρ is an invertible function, then F is single-valued.*

Corollary 2

Let $S \subset T$, $\rho(S) \subset S$, $a, b > 0$, Y *be a real Banach space,* $F: S \to ccl(Y)$

$$pF(x) + qF(y) \subset F(x \star y), \qquad x, y \in S,\ x \star y \in S$$

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$$4F(x+y)+4F(x-y)+24F(x)\subset F(2x+y)+F(2x-y)+6F(y),$$

or

$$F(x+y+z)\subset 2F\left(\frac{x+y}{2}\right)+F(z).$$

Let V be compact and convex subset of a real Banach space Y, $0 \in V$.

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Corollary 3

Let K be a convex cone in a real vector space and $c \in K$. Suppose that $a+b \neq 1$, $p+q>1$ and $f\colon K \to Y$ satisfies

$$f(ax+by+c)-pf(x)-qf(y) \in V, \qquad x,y \in K.$$

Then there exists a unique function $h\colon K \to Y$ such that

$$h(ax+by+c)=ph(x)+qh(y), \qquad x,y \in K,$$

and

$$h(x)-f(x) \in \frac{1}{p+q-1}V, \qquad x \in K.$$

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$$\begin{aligned} F(ax+by+c) &= f(ax+by+c) + \frac{1}{p+q-1}V \\ &\subset pf(x) + qf(y) + \frac{p+q}{p+q-1}V \\ &= p\left(f(x) + \frac{1}{p+q-1}V\right) + q\left(f(y) + \frac{1}{p+q-1}V\right) \\ &= pF(x) + qF(y), \qquad x, y \in K. \end{aligned}$$

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and such that

$$h(ax+by+c) = ph(x) + qh(y), \qquad x, y \in K.$$

Corollary 4

Let $(K,+)$ *be a commutative group and* $f\colon K \to Y$ *satisfies*

$$f(2x+y)+f(2x-y)+6f(y)-4f(x+y)-4f(x-y)-24f(x) \in V$$

for every $x, y \in K$. *Then there exists a unique function* $h\colon K \to Y$ *such that*

$$h(2x+y)+h(2x-y)+6h(y)=4h(x+y)+4h(x-y)+24h(x),$$

$$h(x)-f(x) \in \frac{1}{24}V, \quad x, y \in K.$$

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$$F(2x+y)+F(2x-y)+6F(y) \subset 4F(x+y)+4F(x-y)+24F(x),\ x, y \in K.$$

Theorem 6

Let $F\colon K \to n(Y)$, $k \in \mathbb{N}$, $\Psi\colon K \times Y^k \to Y$, $\alpha_1, \ldots, \alpha_k\colon K \to K$, $\lambda_1, \ldots, \lambda_k\colon K \to [0, \infty)$,

$$d(\Psi(x, w_1, \ldots, w_k), \Psi(x, z_1, \ldots, z_k)) \leq \sum_{i=1}^{k} \lambda_i(x) d(w_i, z_i)$$

for $x \in K$, $w_1, \ldots, w_k, z_1, \ldots, z_k \in Y$ *and*

$$\liminf_{n\to\infty} \sum_{i_1=1}^{k} \lambda_{i_1}(x) \sum_{i_2=1}^{k} (\lambda_{i_2} \circ \alpha_{i_1})(x) \ldots \sum_{i_n=1}^{k} (\lambda_{i_n} \circ \alpha_{i_{n-1}} \circ \ldots \circ \alpha_{i_1})(x) \times \delta(F((\alpha_{i_n} \circ \ldots \circ \alpha_{i_1})(x))) = 0 \qquad \text{for } x \in K.$$

Theorem 6

(1) *If Y is complete and*

$$\Psi(x, F(\alpha_1(x)), \ldots, F(\alpha_k(x))) \subset F(x), \qquad x \in K,$$

then there exists a unique selection $f : K \to Y$ of the multifunction $\mathrm{cl}\, F$ such that $\Psi(x, f(\alpha_1(x)), \ldots, f(\alpha_k(x))) = f(x)$ for $x \in K$.

(2) *If*

$$F(x) \subset \Psi(x, F(\alpha_1(x)), \ldots, F(\alpha_k(x))), \qquad x \in K,$$

then F is a single-valued function and $\Psi(x, F(\alpha_1(x)), \ldots, F(\alpha_k(x))) = F(x)$ for $x \in K$.

Corollary 5

Let $F\colon K \to n(Y)$, $a\colon K \to K$, $\phi\colon K \to \mathbb{R}$ *and*

$$\liminf_{n\to\infty} |\phi(x)||\phi(a(x))| \ldots |\phi(a^{n-1}(x))|\delta(F(a^n(x))) = 0.$$

(1) *If* Y *is complete and*

$$\phi(x)F(a(x)) \subset F(x), \qquad x \in K,$$

then there exists a unique selection $f\colon K \to Y$ *of the multifunction* $\mathrm{cl}\, F$ *such that* $\phi(x)f(a(x)) = f(x)$ *for* $x \in K$.

(2) *If*

$$F(x) \subset \phi(x)F(a(x)), \qquad x \in K,$$

then F *is single-valued function and* $\phi(x)F(a(x)) = F(x)$ *for* $x \in K$.

Corollary 6

Let $F\colon K \to n(Y)$, $k \in \mathbb{N}$, $\Psi\colon K \times Y^k \to Y$, $\alpha_1, \ldots, \alpha_k\colon K \to K$, $\lambda_1, \ldots, \lambda_k \in \mathbb{R}_+$, $\lambda_1 + \ldots + \lambda_k \in (0,1)$, $M = \sup\{\delta(F(x)) :\ x \in K\} < \infty$.

(1) *If* Y *is complete and*

$$\lambda_1 F(\alpha_1(x)) + \ldots + \lambda_k F(\alpha_k(x)) \subset F(x), \qquad x \in K,$$

then there exists a unique selection $f\colon K \to Y$ *of the multifunction* $\mathrm{cl}\, F$ *such that* $\lambda_1 f(\alpha_1(x)) + \ldots + \lambda_k f(\alpha_k(x)) = f(x)$ *for* $x \in K$.

(2) *If*

$$F(x) \subset \lambda_1 F(\alpha_1(x)) + \ldots + \lambda_k F(\alpha_k(x)), \qquad x \in K,$$

then F *is single-valued function and* $\lambda_1 F(\alpha_1(x)) + \ldots + \lambda_k F(\alpha_k(x)) = F(x)$ *for* $x \in K$.

- *The properties of functional inclusions and Hyers-Ulam stability*, Aequationes Math. 85 (2013), 111–118.
- *On selections of set-valued inclusions in a single variable with applications to several variables*, Results Math. 64 (2013), 1–12.
- *On selections of set-valued maps satisfying some inclusions in a single variable*, Math. Slovaca, to apear.