# Extensions of functions with a closed graph

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Stara Lesna, 1-5 September 2014

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- $D(f)$ – the set of all discontinuity points of $f$,
- $G(f)$ – the graph of $f$:

$$G(f) = \{(x, y) : y = f(x)\};$$

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- $\mathcal{F}^+(X)$ – the family of all nonnegative functions defined on $X$.

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$$\mathcal{B}_1(X) \supsetneq \mathcal{P}(X) \supsetneq \mathcal{P}_0(X) \supsetneq \mathcal{U}(X) \supsetneq \mathcal{C}(X).$$

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Let $L$ be a cone in $\mathbb{R}^\Gamma$ (i.e. $L + L \subset L$, $aL \subset L$ for every $a \in \mathbb{R}^+$, and $L \cap (-L) = \{0\}$). We say that a mapping $f\colon L \to \mathbb{R}$ is **positively affine** if, for any elements $x_1, x_2 \in L$ and $a, b \in \mathbb{R}_+$ such that $a + b = 1$, we have $f(ax_1 + bx_2) = af(x_1) + bf(x_2)$.

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## Borsuk-Dugundji, 1933-1951

Let $A$ be a closed subset of a metric space $X$. Then there is a linear extension operator from $\mathcal{C}(A)$ into $\mathcal{C}(X)$. The restriction of this operator to the space of all bounded elements of $\mathcal{C}(A)$ is a positive isometry into $\mathcal{C}^b(X)$.

# Historical results

**Kalenda-Spurný, 2005**

Let $Y$ be a Lindelöf hereditarily Baire subset of a completely regular space $X$ and let $f$ be a Baire-one function on $Y$. Then there is a Baire-one function $g$ on $X$ such that $f = g$ on $Y$, $\inf f(Y) = \inf g(X)$ and $\sup f(Y) = \sup g(X)$.

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## Wójtowicz-Sieg, 2010

If $X$ is a $P$-space (i.e., every $G_\delta$-subset of $X$ is open) then for every closed subset $A$ of $X$, every $f_0 \in \mathcal{U}(A)$ can be extended to $f \in \mathcal{U}(X)$.

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Let $X$ be a Hausdorff topological space, let $A$ be a zero-set (i.e. $A = [g = 0] := g^{-1}(0)$ for some $g \in \mathcal{C}(X)$), and let $f_0 \colon A \to \mathbb{R}$ be a function with a closed graph.

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The symbol $f_{(A,g)}$ denotes a real function defined on $X$ by the formula

$$f_{(A,g)}(x) = \begin{cases} f_0(x) & , \quad x \in A, \\ \frac{1}{g(x)} & , \quad x \notin A. \end{cases} \tag{1}$$

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The symbol $\mathrm{Ext}_{(A,g)}$ denotes the mapping $\mathrm{Ext}_{(A,g)} \colon \mathbb{R}^A \to \mathbb{R}^X$ defined by the formula

$$\mathrm{Ext}_{(A,g)}(f) = f_{(A,g)}.$$

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(a) there is a function $f\colon X \to \mathbb{R}$ with a closed graph such that $f_{\restriction A} = f_0$, and

(b) the set $D(f)$, of points of discontinuity of $f$, is of the form

$$D(f) = D(f_0) \cup \operatorname{bd} A. \tag{2}$$

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More exactly, for every fixed function $g \in \mathcal{C}(X)$ such that $A = g^{-1}(0)$, the operator $\mathrm{Ext}_{(A,g)}$ defined above maps $\mathcal{U}(A)$ into $\mathcal{U}(X)$ and is positively affine.

# First main result

**Corollary**

Let $A$ be a closed and $G_\delta$ (closed, respectively) subset of a normal (perfectly normal, respectively) space $X$. Then there is a positively affine extension operator $\mathrm{Ext}\colon \mathcal{U}(A) \to \mathcal{U}(X)$.

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The symbol $\widetilde{f}$ denotes a real function defined on $X$ by the formula

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{dla} \quad x \in A \\ 0 & \text{dla} \quad x \notin A. \end{cases}$$

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The symbol $T$ denotes the mapping $T\colon \mathcal{P}_0(A) \to \mathbb{R}^X$ defined by the formula $T(f) = \widetilde{f}$.

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Let $X$ be a normal topological space, and let $A$ be a closed and $G_\delta$ subset of $X$. Let also $f \in \mathcal{P}_0(A)$. Then

1. the counterdomain of $T$ is $\mathcal{P}_0(X)$,
2. $T$ is a linear extension operator from $\mathcal{P}_0(A)$ into $\mathcal{P}_0(X)$ such that its restriction to the class $\mathcal{P}_0^b(A)$ is an isometry (i.e., $\|T(f)\|_X = \|f\|_A$).

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**Thank You very much for Your attention :-)**