# Generalizations of Urysohn's lemma for the family of extra strong Świątkowski functions

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# Introduction

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function.

**Definition**

We say that $f$ is *Darboux*, if it maps connected sets onto connected sets; i.e., if whenever $a < b$ and $y$ is a number between $f(a)$ and $f(b)$, there is $x_0 \in (a, b)$ such that $f(x_0) = y$.

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We say that $f$ is *quasi-continuous* in the sense of Kempisty, if for all $x \in \mathbb{R}$ and open sets $U \ni x$ and $V \ni f(x)$, the set $\operatorname{int}(U \cap f^{-1}(V)) \neq \emptyset$.

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We say that $f$ is *strong Świątkowski*, if whenever $a < b$ and $y$ is a number between $f(a)$ and $f(b)$, there is $x_0 \in (a, b) \cap \mathcal{C}(f)$ such that $f(x_0) = y$.

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The symbols below denote the following families of real functions:

- $\mathcal{C}$ – the class of all continuous functions,
- $\mathcal{D}$ – the class of all Darboux functions,
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It can be readily verified that we have the following proper inclusions:

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We say that a set $A \subset \mathbb{R}$ is *semi-open*, if $A \subset \operatorname{cl} \operatorname{int} A$, and it is *semi-closed*, if $\operatorname{int} \operatorname{cl} A \subset A$.

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Notice that a set $A \subset \mathbb{R}$ is semi-open if and only if its complement is semi-closed.

# Classical separation property

**Lemma (Urysohn 1925)**

*Let $X$ be a normal topological space and sets $A_0, A_1 \subset X$ be disjoint and closed. Then there exists a continuous function $f\colon X \to \mathbb{R}$ such that $f = 0$ on $A_0$ and $f = 1$ on $A_1$. Moreover, if $A_0$ and $A_1$ are $G_\delta$-sets, then we can require that $f(x) \in (0, 1)$ for each $x \in X \setminus (A_0 \cup A_1)$.*

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The symbol $[f = a]$ stands for the set $\{x \in \mathbb{R}\ ;\ f(x) = a\}$.

**Theorem**

*Let $X$ be a perfectly normal topological space and sets $A_0, A_1 \subset X$ be disjoint. There is a continuous function $f\colon X \to \mathbb{R}$ such that $A_0 = [f = 0]$ and $A_1 = [f = 1]$ if and only if sets $A_0$ and $A_1$ are closed.*

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**Theorem (Maliszewski 2002)**

*Let $A_0, A_1 \subset \mathbb{R}$ be disjoint. There is a Darboux function $f\colon \mathbb{R} \to \mathbb{R}$ such that $A_0 = [f = 0]$ and $A_1 = [f = 1]$ if and only if the set $\mathbb{R} \setminus (A_0 \cup A_1)$ is bilaterally $\mathfrak{c}$-dense in itself and $\operatorname{card}\big(\mathsf{I}[\alpha, \beta] \setminus (A_0 \cup A_1)\big) = \mathfrak{c}$ for each $\alpha \in A_0$ and $\beta \in A_1$.*

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**Theorem (Kowalewski 2005)**

*Let $A_0, A_1 \subset \mathbb{R}$ be disjoint. There is a quasi-continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that $A_0 = [f = 0]$ and $A_1 = [f = 1]$ if and only if sets $A_0$, $A_1$ and $A_0 \cup A_1$ are semi-closed.*

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**Theorem (Szczuka 2014)**

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*Let $X$ be a perfectly normal topological space and sets $A^+, A^- \subset X$ be disjoint. There is a continuous function $f\colon X \to \mathbb{R}$ such that $A^+ = [f > 0]$ and $A^- = [f < 0]$ if and only if sets $A^+$ and $A^-$ are open.*

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Characterize a pair $(A^+, A^-)$ of disjoint sets for which there is an extra strong Świątkowski function $f\colon \mathbb{R} \to \mathbb{R}$ such that $A^+ = [f > 0]$ and $A^- = [f < 0]$.

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