Periodic points of some maps of Jordan domains

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28th Summer School on Real Function Theory
Stará Lesná
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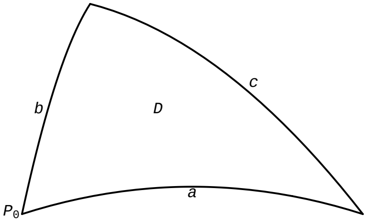
Notation

Let D be a closed domain in the plain the boundary of which consists of three arcs a, b and c.



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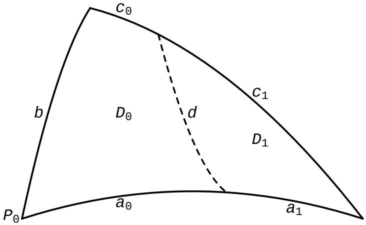

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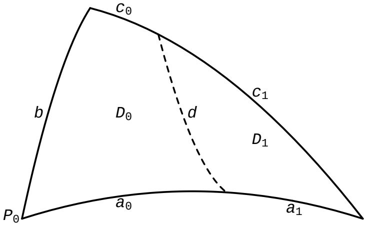

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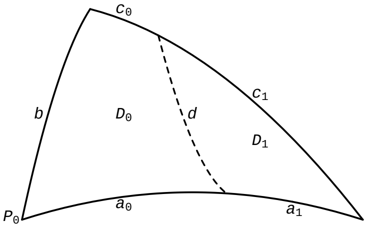

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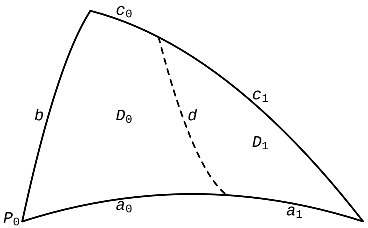

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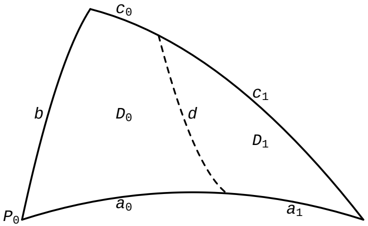

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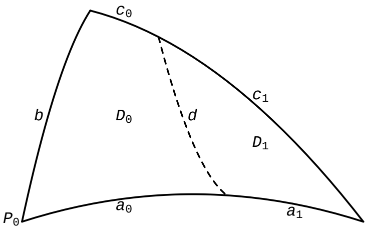

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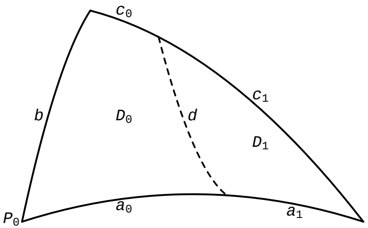

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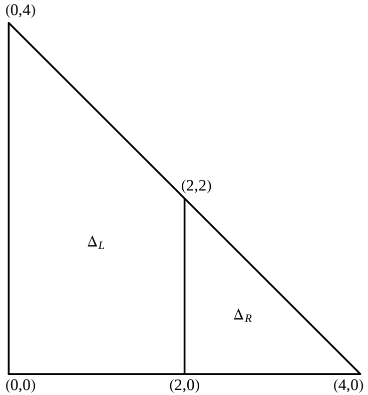

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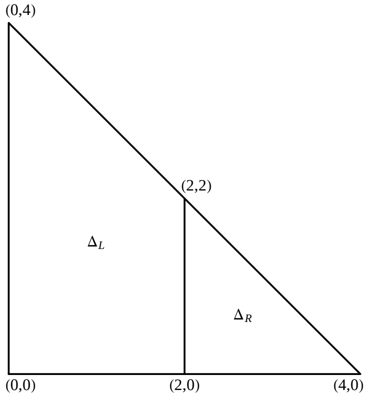

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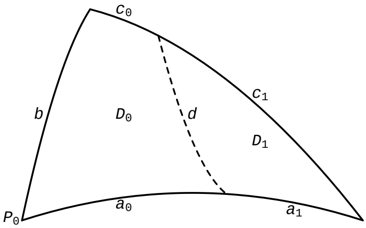

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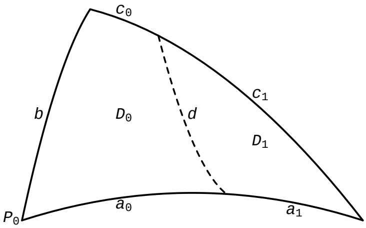

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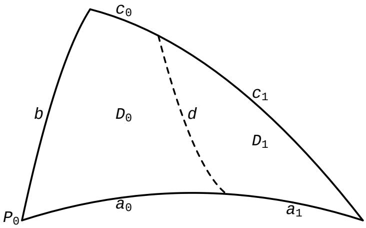

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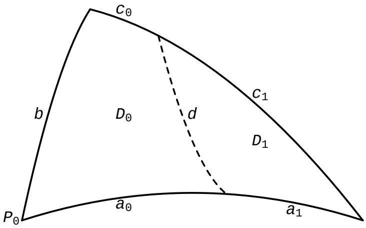

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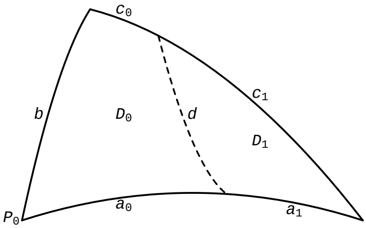

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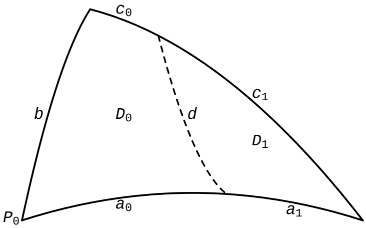

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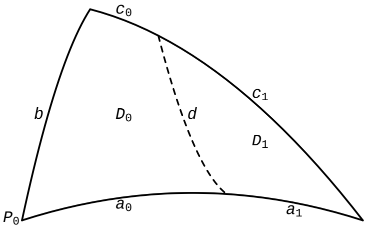

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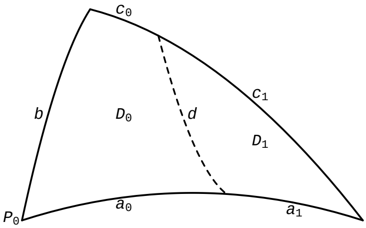

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For a periodic point P of the map G we consider its itinerary W as a sequence $(w_i)_{i=0}^{\infty}$ defined by

$$w_i = \begin{cases} 0 & \text{if } G^i(P) \in D_0, \\ 1 & \text{if } G^i(P) \in D_1. \end{cases}$$

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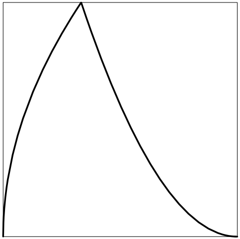

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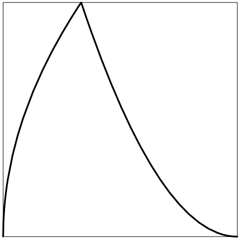

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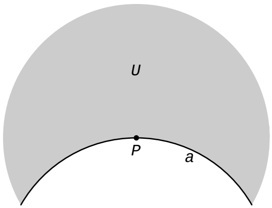

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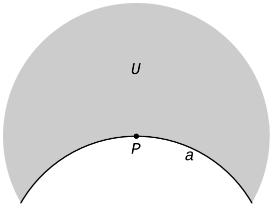

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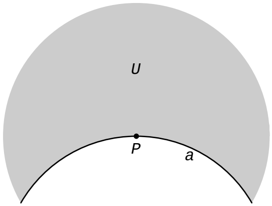

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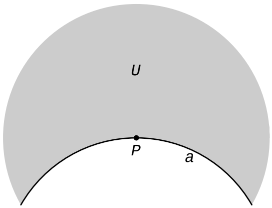

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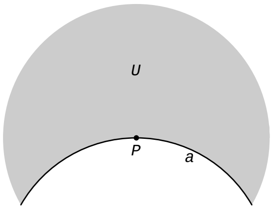

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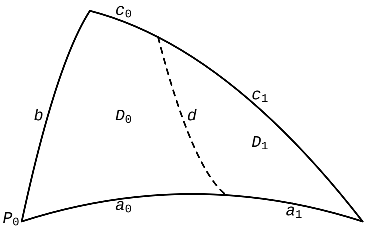

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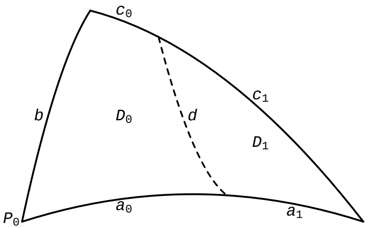

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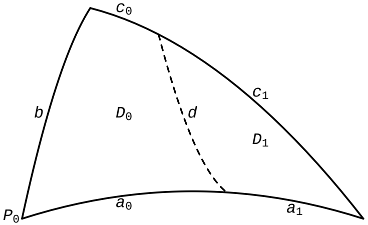

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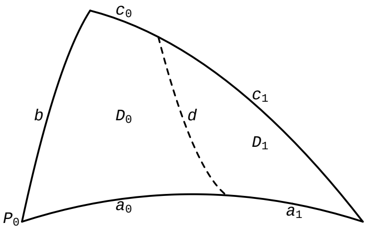

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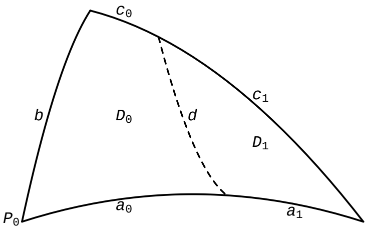

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Theorem

Let $P \in a$, $P \neq P_0$ be a periodic point of the map G with period n such that

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Jacobi matrix

Let $P=(x_0,0)\in\Delta$ be a fixed point of the map F^n. In this case $P=\left(4\sin^2\frac{k\pi}{2^n\pm 1},0\right)$. Then the Jacobi matrix of the map F^n at the point P has a form

$$\begin{pmatrix}\lambda_1 & \mu\\ 0 & \lambda_2\end{pmatrix}=\begin{pmatrix}\mp 2^n & \mu\\ 0 & \prod\limits_{i=0}^{n-1}4\sin^2\frac{2^ik\pi}{2^n\pm 1}\end{pmatrix}.$$

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Classification

For λ_2 we have the possibilities

Saddle point

$0 \leq \lambda_2 < 1$, e.g. $x_0 = 4\sin^2 \frac{\pi}{17}$

Nonhyperbolic point

$\lambda_2 = 1$, e.g. $x_0 = 4\sin^2 \frac{\pi}{15}$

Repulsive point

$1 < \lambda_2$, e.g. $x_0 = 4\sin^2 \frac{3\pi}{17}$

Remark

All above points $(x_0, 0)$ have period 4.

Classification

Saddle point

Lower periodic points with period n and $0 < \lambda_2 < 1$ appear for all $n \geq 4$.

Nonhyperbolic point

Lower periodic points with period n and $\lambda_2 = 1$ appear for infinitely many n, e.g. $n = 4 \cdot 3^i \cdot 5^j$, where $i \geq 0$, $j \geq 0$.

Repulsive point

Lower periodic points with period n and $1 < \lambda_2$ appear for all $n \geq 1$.

Modifications

Assume that for any $x \in (0,\ 4)$ we have an increasing homeomorphism φ_x of the interval $\langle 0,\ 4-x \rangle$ onto itself. Moreover let the function $\varphi(x,y) = \varphi_x(y)$ be continuous in the domain

$$\widehat{\Delta} = \{\, (x,y) : 0 < x < 4\,, 0 \le y \le 4-x \}\ .$$

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To obtain such above family φ_x, choose for $0 < x < 4$ a family of increasing homeomorphisms ψ_x of the interval $\langle 0,\ 1\rangle$ such that the function $\psi(x, y) = \psi_x(y)$ is continuous in $(0, 4) \times \langle 0,\ 1\rangle$ and put

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$$\widetilde{\lambda}_2 = \prod_{i=0}^{n-1} x_i \varphi'_{x_i}(0) = \prod_{i=0}^{n-1} x_i \psi'_{x_i}(0)\ ,$$

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- F. Balibrea, J. L. G. Guirao, M. Lampart and J. Llibre. Dynamics of a Lotka–Volterra map. *Fund. Math.* **191** (2006), 265–279.
- J. L. G. Guirao and M. Lampart. Transitivity of a Lotka–Volterra map. *Discrete Contin. Dyn. Syst. Ser. B* **9** (2008), no. 1, 75–82 (electronic).
- P. Maličký. Interior periodic points of a Lotka–Volterra map. *J. Difference Equ. Appl.* **18** (2012), no. 4, 553–567.
- A. N. Sharkovskiĭ. Low dimensional dynamics. *Tagunsbericht 20/1993, Proceedings of Mathematisches Forschunginstitut Oberwolfach* 1993, 17.
- G. Swirszcz. On a certain map of the triangle. *Fund. Math.* **155** (1998), 45–57.