On M-sets for series with respect to characters of compact zero-dimensional groups

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Survey: A. S. Kechris, A. Louveau , Descriptive set theory and the structure of sets of uniqueness, London Mathematical Society lecture series 128, Cambridge University Press, 1987.

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Then the order of G_0/G_1 is p_0, and the order of G_0/G_n, $n = 1, 2, ...$, is

$$m_n := p_0 \cdot p_1 \cdot ... \cdot p_{n-1},$$

with $p_i \geq 2$ for all i (put $m_0 := 1$).

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The group Γ is discrete and it can be represented as a sum of increasing chain of finite subgroups

$$\Gamma_0 \subset \Gamma_1 \subset \Gamma_2 \subset \ldots \subset \Gamma_n \subset \ldots$$

where $\Gamma_0 = \{\gamma^{(0)}\}$ with $\gamma^{(0)}(g) = 1$ for all $g \in G$.

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where $\Gamma_0 = \{\gamma^{(0)}\}$ with $\gamma^{(0)}(g) = 1$ for all $g \in G$.

For each n the group Γ_n is the annihilator of G_n, i.e.,

$$\Gamma_n = G_n^{\perp} := \{\gamma \in \Gamma : (\gamma(g) = 1 \;\; \text{for all} \;\; g \in G_n\}.$$

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with respect to this system.

Important subsequence of partial sums of this series are

$$\sum_{\gamma \in \Gamma_n} a_\gamma \gamma.$$

Properties of the characters

For each coset K_n of G_n choose and fix an element g_{K_n}. Then for each $n \in \mathbb{Z}$ we can represent any element $g \in G$ in the form:

$$g = g_{K_n} + \{g\}_n$$

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Lemma

If $\gamma \in \Gamma_n$ then γ is constant on each coset K_n of G_n.

Numeration of the characters

The factor groups $\Gamma_{n+1}/\Gamma_n = G_{n+1}^{\perp}/G_n^{\perp}$ and G_n/G_{n+1} are isomorphic and so they are of the same order p_n for each $n \in \mathbb{N}$. This implies that the group Γ_n has $m_n = p_0 \cdot p_1 \cdot \ldots \cdot p_{n-1}$ elements.

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For

$$n = \sum_{k=0}^{s} t_k m_k, \quad 0 \le t_k \le p_k - 1$$

define

$$\gamma_n := \prod_{k=0}^{s} (\gamma_{m_k})^{t_k}.$$

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For example for $p = 2$ $\quad \gamma_{2^2}$ assumes the values

$$1, \quad -1, \quad i, \quad -i, \quad \exp(\frac{\pi i}{4}), \quad -\exp(\frac{\pi i}{4}), \quad \exp(\frac{3\pi i}{4}), \quad -\exp(\frac{3\pi i}{4}).$$

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For $m = \sum_{n=0}^{n_m} t_n p^n$ $\quad \gamma_m$ is a product of γ_{p^n}.

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$$S_{m_n}(g) = \frac{1}{|K_n|} \int_{K_n} S_{m_n} d\mu = \frac{\psi(K_n)}{|K_n|} \tag{1}$$

for any point $g \in K_n$.

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So at any point $g \in G$ we get

$$\lim_{n\to\infty} S_{m_n}(g) = D_{\mathcal{B}}\psi(g).$$

QUASI-MEASURE

Theorem

Any series w.r. to Γ is Fourier-Stieltjes series w.r. to quasi-measure, so that

$$a_\gamma = \hat{\psi}(\gamma) = \int_G \gamma d\psi.$$

A set $E \subset G$ is M-set w.r. to Γ iff there exists a μ-singular quasi-measure ψ supported by E with

$$\lim_{n\to\infty} \hat{\psi}(n) = 0.$$

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Theorem

Let some integration process $\mathcal{A}$ be given which produces an integral additive on $\mathcal{I}$. Let a $\mathcal{B}$-interval function ψ be the quasi-measure generated by the series and (1) holds. Then this series is the Fourier series of an $\mathcal{A}$-integrable function f if and only if $\psi(I) = (\mathcal{A}) \int_I f$ for any $\mathcal{B}$-interval I.

Localization theorem

Localization Theorem.

Let the series

$$\sum_n c_n \gamma_n \tag{2}$$

is a formal product of the series

$$\sum_n a_n \gamma_n, \quad a_n \to 0,$$

and a polynomial $P = \sum_{n=0}^{k} b_n \gamma_n$. Then the series (2) and $P \sum_n a_n \gamma_n$ are uniformly equiconvergent, i.e., their difference is a series that is uniformly convergent on G with sum zero.

Localization theorem

Corollary.

For a series

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and a coset K_n there exists a series

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Lemma.

If for the series $\sum_n a_n \gamma_n$ with respect to the system Γ satisfying the condition $lim_{n\to\infty} a_n = 0$, some subsequence of partial sums of the form $S_{m_{n_k}}(g)$ converges to zero for all g in some open set O, then this series converges to zero on O.

Construction of M_0-set

Let h be a nondecreasing, right-continuous function, $h(0) = 0$. The Hausdorff h-measure $\mathcal{H}^h$ is defined by

$$\mathcal{H}^h(A) = \liminf_{\delta \to 0} \sum \{h(diam(E_i)) : E_i \text{ open}, \cup E_i \supset A, \ diam(E_i) \le \delta\}.$$

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Theorem

For any h there exists a perfect M-set E for the system Γ whose Hausdorff h-measure equals zero. Moreover it is M_0-set, i.e., a set of *strict multiplicity* (corresponding quasi-measure is a probability measure concentrated on E).

Construction of M-set

Lemma (main step in construction).

For any $A > 0$, $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for any K_n, $n \geq N$, any $\delta > 0$, B, $0 < |B| \leq A$ and any natural λ there exists a polynomial

$$P(g) = \sum_{j=m_\lambda}^{m_l - 1} a_j \gamma_j(g)$$

with properties:
1. $|a_j| < \varepsilon, \quad m_\lambda \leq j \leq m_l - 1;$
2. $P(g) = 0$ if $g \in G \setminus K_n;$
3. $|P(g)| \geq |B|$ if $g \in K_n;$
4. $sh(1/m_l) < \delta$ where s is the number of those K_l in K_n for which $|P(g)| \neq |B|$ with $g \in K_l$.

Construction of M-set

Now using this Lemma we construct by induction a sequence of polynomial Q_k and a sequence of sets E_k, $E_{k-1} \subset E_k$, such that

$$\sum_{j=1}^{k} Q_j(t) = 0$$

outside E_{k-1} and $\mu_h E = 0$ where $E = \cap_{k=1}^{\infty} E_k$.

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It can be proved that required null-series is

$$\sum_{j=1}^{\infty} Q_j(t)$$

and E is required M_0-set.