Some problems of representation of functions by multiple Haar and Walsh series and generalized integrals

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Walsh functions

The most part of this talk is related to the paper

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V. Skvortsov, F. Tulone

Multidimensional dyadic Kurzweil-Henstock- and Perron-type integrals in the theory of Haar and Walsh series (to be published in JMAA).

1. Introduction
   - Motivation I: Recovering coefficients for multidimensional convergent series
   - Motivation II: Suitable method of integration

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   - Motivation II: Suitable method of integration
2. The Perron-type Integrals
   - The P_d-integral
   - Contr-example
   - The P_dS-integral

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Put $\chi_0(x) \equiv 1$. If $n = 2^k + i - 1$, $k = 0, 1, \ldots$, $i = 1, 2, \ldots, 2^k$, we put

$$\chi_n(x) := \begin{cases} 2^{k/2}, & \text{if } x \in \left(\frac{2i-2}{2^{k+1}}, \frac{2i-1}{2^{k+1}}\right), \\ -2^{k/2}, & \text{if } x \in \left(\frac{2i-1}{2^{k+1}}, \frac{2i}{2^{k+1}}\right), \\ 0, & \text{if } x \in (0,1) \setminus \left[\frac{2i-2}{2^{k+1}}, \frac{2i}{2^{k+1}}\right], \end{cases}$$

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and we agree that at each point of discontinuity

$\chi_n(x) = \frac{1}{2}(\chi_n(x+0) + \chi_n(x-0))$ and that at $x = 0$ and $x = 1$ Haar functions are continuous from the right and from the left, respectively.

Walsh functions

The Rademacher functions $r_n,\ n = 0, 1, \ldots,$ on $[0, 1]$ is

$$r_n(x) = sign\,\sin(2^{n+1})\pi x \quad \text{if} \quad x \in (0, 1),$$

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If $n = \sum_{i=0}^{\infty} n_i 2^i,$ with $n_i \in \{0, 1\}$, is the dyadic representation of $n \geq 0$, we put

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$$\sum_{\mathbf{n}=\mathbf{0}}^{\infty} a_{\mathbf{n}} w_{\mathbf{n}}(\mathbf{x}) := \sum_{n_1=0}^{\infty} \dots \sum_{n_m=0}^{\infty} a_{n_1,\dots,n_m} \prod_{i=1}^{m} w_{n_i}(x_i) \quad (1)$$

$$\sum_{\mathbf{n}=\mathbf{0}}^{\infty} b_{\mathbf{n}} \chi_{\mathbf{n}}(\mathbf{x}) := \sum_{n_1=0}^{\infty} \dots \sum_{n_m=0}^{\infty} b_{n_1,\dots,n_m} \prod_{i=1}^{m} \chi_{n_i}(x_i) \quad (2)$$

where $a_{\mathbf{n}}$ and $b_{\mathbf{n}}$ are real numbers and $\mathbf{n} = (n_1, ..., n_m)$.

m-dimensional dyadic interval

Let Q_d be the set of all dyadic-rational numbers in $[0,1]$, i.e., the numbers of the form $\frac{j}{2^n}$ with $0 \leq j \leq 2^n, \quad n = 0,1,2,\ldots$. The points $[0,1] \setminus Q_d$ constitute the set of dyadic-irrational numbers in $[0,1]$.

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Let $\mathcal{I}$ be the family of all m-dimensional dyadic intervals

$$I_{\mathbf{j}}^{(\mathbf{n})} := I_{j_1}^{(n_1)} \times \cdots \times I_{j_m}^{(n_m)} \tag{3}$$

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We denote by $I^{(\mathbf{n})}$ an arbitrary interval of rank $\mathbf{n}$.

Some remarks on m-dimensional dyadic interval

If $\mathbf{x} = \{x_1, ..., x_m\} \in K$ there exists a sequence of m-dimensional dyadic intervals $\{I_{\mathbf{x}}^{(\mathbf{n})}\}$ such that $\cap_{\mathbf{n}} I_{\mathbf{x}}^{(\mathbf{n})} = \{\mathbf{x}\}$.

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It follows from the definitions of Wash and Haar function that for $\mathbf{n} = (n_1, ..., n_m)$ with $2^{k_j - 1} \leq n_j < 2^{k_j}$, $j = 1, ..., m$, the functions $\chi_{\mathbf{n}}$ and $w_{\mathbf{n}}$ are constant in the interior of each dyadic interval of rank $\mathbf{k} = (k_1, ..., k_m)$.

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Moreover, with the same notation, the functions $\chi_{\mathbf{n}}$ are supported by some intervals of rank $\mathbf{k} - \mathbf{1} = (k_1 - 1, ..., k_m - 1)$.

Nth-partial sum

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N-th rectangular partial sum

If $\mathbf{N} = (N_1, \ldots, N_m)$, then the Nth rectangular partial sums $S_{\mathbf{N}}$ of series (1) and (2) at a point $\mathbf{x} = (x_1, \ldots, x_m)$ are

$$S_{\mathbf{N}}(\mathbf{x}) := \sum_{n_1=0}^{N_1-1} \ldots \sum_{n_m=0}^{N_m-1} a_{\mathbf{n}} w_{\mathbf{n}}(\mathbf{x})$$

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Rectangular and Regular of convergence

Rectangular convergence

The Walsh and Haar series rectangularly converges to sum $S(\mathbf{x})$ at point $\mathbf{x}$ if

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Regular convergence

Let $\rho \in (0, 1]$; then the Walsh and Haar series ρ-regularly converges to sum $S(\mathbf{x})$ at a point $\mathbf{x}$ if

$$S_{\mathbf{N}}(\mathbf{x}) \to S(\mathbf{x}) \text{ as } \min_i \{N_i\} \to \infty \text{ and } reg(\mathbf{N}) \geq \rho.$$

Quasi measure generated by series

We define an interval function $\psi(I)$ on $\mathcal{I}$ by $\psi(I_{\mathbf{j}}^{(\mathbf{k})}) := \int_{I_{\mathbf{j}}^{(\mathbf{k})}} S_{2^{\mathbf{k}}}$ where $2^{\mathbf{k}}$ stand for $(2^{k_1}, \ldots, 2^{k_m})$.

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We note that if $\mathbf{k_1} > \mathbf{k}$, in the sense of the coordinate-wise inequality, then

$$\int_{I_{\mathbf{j}}^{(\mathbf{k})}} S_{2^{\mathbf{k_1}}} = \int_{I_{\mathbf{j}}^{(\mathbf{k})}} S_{2^{\mathbf{k}}}. \tag{4}$$

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In dyadic analysis the function ψ is referred to as the quasi-measure generated by the series.

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for any point $\mathbf{x} \in int(I_{\mathbf{j}}^{(\mathbf{k})})$.

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The equality (5) gives a relation between convergence of partial sums of the series and differentiability of ψ with respect to the basis of dyadic intervals at $\mathbf{x}$. The type of convergence of the series implies the corresponding type of derivative.

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Considering various types of limits on the right side of (5) we get corresponding type of the derivative with respect to dyadic system.

d-derivative

In particular we will use the following D_d-derivative

Definition (d-derivative)

Given a function F defined on $\mathcal{I}$, $\mathbf{k} = (\mathbf{k_1}, ..., \mathbf{k_m})$ the rank of the m-dimensional interval $I_{\mathbf{x}}^{(\mathbf{k})}$, the upper and the lower d-derivatives of F at a point $\mathbf{x}$, with respect to the dyadic system, are defined as

$$\overline{D}_d F(\mathbf{x}) := \limsup_{min_j k_j \to \infty} \frac{F(I_{\mathbf{x}}^{(\mathbf{k})})}{|I_{\mathbf{x}}^{(\mathbf{k})}|} \quad \text{and} \quad \underline{D}_d F(\mathbf{x}) := \liminf_{min_j k_j \to \infty} \frac{F(I_{\mathbf{x}}^{(\mathbf{k})})}{|I_{\mathbf{x}}^{(\mathbf{k})}|},$$

respectively. If $\overline{D}_d F(\mathbf{x}) = \underline{D}_d F(\mathbf{x})$ we call this common value the d-derivative $D_d F(\mathbf{x})$ at $\mathbf{x}$. We say that F is d-differentiable at $\mathbf{x}$ if the d-derivative at this point exists and is finite.

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If in the previous definition of d-derivative we consider limits with respect the intervals being ρ-regular then we get the definition of ρ-regular d-derivative. Using other types of the limits we get other derivatives (for example the so called ordinary derivative)

Quasi-measure and derivatives

At least at the points $\mathbf{x}$ with all coordinates being dyadic irrational, we have

Convergence and differentiability

$$\lim_{\mathbf{k}\to\infty} S_{2^{\mathbf{k}}}(\mathbf{x}) = D_d\psi(\mathbf{x}), \tag{6}$$

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This limit allow to reduce some problem on the convergence of series to the corresponding problem of differentiability and viceversa.

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Proposition 1

Let some integration process $\mathcal{A}$ be given which produces an integral additive on $\mathcal{I}$. Assume a Walsh series or Haar series is given. Let a function ψ defined on $\mathcal{I}$ be the quasi-measure generated by this series and (5) holds. Then this series is the Fourier series of an $\mathcal{A}$-integrable function f if and only if $\psi(I) = (\mathcal{A}) \int_I f$ for any $I \in \mathcal{I}$.

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Proposition 1

Let some integration process $\mathcal{A}$ be given which produces an integral additive on $\mathcal{I}$. Assume a Walsh series or Haar series is given. Let a function ψ defined on $\mathcal{I}$ be the quasi-measure generated by this series and (5) holds. Then this series is the Fourier series of an $\mathcal{A}$-integrable function f if and only if $\psi(I) = (\mathcal{A}) \int_I f$ for any $I \in \mathcal{I}$.

The formula (5) and the proposition above give a simple method of summation of any Walsh- or Haar-Fourier series of function f. This result is useful even in one-dimensional case because Walsh-Fourier-Lebesgue series can be divergent almost everywhere (example analogous to Kolmogorov example)

Remark on uniqueness of solution

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The convergence of series everywhere in formulation of the coefficients problem can be replaced by convergence everywhere outside some particular exceptional sets, so-called sets of uniqueness or U-sets. We recall that a set E is said to be U-set for a system of functions if the convergence of a series with respect to this system to zero outside the set E implies that all coefficients of the series are zero.

Remark on uniqueness of solution

For multidimensional Walsh series and rectangular convergence it was proved by Skvortsov that the following set Z of points having at least one dyadic-rational coordinate, i.e.,

$$Z := \bigcup_{i=1}^{m} ([0,1]^{i-1} \times Q_d \times [0,1]^{m-i}).$$

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In view of (6) and the above Proposition 1, in order to solve the coefficient problem it is enough to show that the quasi-measure ψ generated by Walsh series is the indefinite integral of its d-derivative which exists at least on $K \setminus Z$.

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In view of (6) and the above Proposition 1, in order to solve the coefficient problem it is enough to show that the quasi-measure ψ generated by Walsh series is the indefinite integral of its d-derivative which exists at least on $K \setminus Z$.

By this we reduce the problem of recovering the coefficients to the one of recovering the primitive and we can use the corresponding theorem on primitives.

Remark on uniqueness of solution

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It can be shown that usual continuity with respect to the dyadic basis (we shall call it d-continuity) is not enough for this purpose and we introduce a stronger notion of continuity, which we shall call local Saks continuity (shortly dS-continuity) with respect to the basis.

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A usual way to solve the problem of primitive is to use a Kurzweil-Henstock-type integral. Unfortunately, in contrast with the one dimensional case, the exceptional set where the derivative of the quasi-measure related to the convergent series is not defined, is not countable in dimension greater than one.

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Because of it the Kurzweil-Henstock-type approach does not work. So we shall define, for our goal a Perron-type integral based on dS-continuity.

d-Continuity

Definition (Continuity in dyadic setting)

We say that a set-function F defined on $\mathcal{I}$, is continuous at a point $\mathbf{x}$, with respect to the m-dimensional dyadic intervals, briefly d-continuous, if for any sequence of m-dimensional dyadic intervals $I_{\mathbf{x}}^{(\mathbf{n})}$ containing $\mathbf{x}$, the value of function F on these intervals tends to zero together with diameter of the intervals.

d-Continuity in sense of Saks

We recall that an interval function F is said to be continuous in the sense of Saks if $\lim_{|I|\to 0,} F(I) = 0$. We define a local version of such a continuity applied to m-dimensional dyadic setting.

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In the two-dimensional case the last equality can be rewritten in terms of ranks of m-dimensional dyadic intervals in the following way:

$$\lim_{k+l\to\infty} F(I_{\mathbf{x}}^{(k,l)}) = 0.$$

dS-continuity of quasi measure

We deduce dS-continuity of quasi-measure from some properties of coefficients of the series. To simplify calculation, we shall formulate most of the results for the two-dimensional case, but all of them are true for any dimension.

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Lemma (dS-continuity of quasi-measure for Wash case)

If a two-dimensional Walsh series is rectangular convergent on the "cross" $\{a \times [0,1]\} \cup \{[0,1] \times b\}$, where $a, b \notin Q_d$, everywhere except a countable set, then at each point $(x,y) \in K$ the quasi-measure ψ is dS-continuous, i.e., $\lim_{|I| \to 0} F(I_{\mathbf{x}}) = 0$ everywhere on K.

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Lemma (dS-continuity of quasi-measure Haar case)

If a two-dimensional Haar series is rectangular convergent everywhere on the unit square K, then at each point $(x, y) \in K$ the quasi-measure ψ is dS-continuous, i.e., $\lim_{|I|\to 0} F(I_{\mathbf{x}}) = 0$ at $\mathbf{x} = (x, y)$.

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Note that the ρ-regular convergence, even everywhere on K does not imply dS-continuity of the corresponding quasi-measure (Plotnikov 2007).

Perron type approach

Definition (dS-major and dS-minor functions)

Let f be a point-function defined at least on $K \setminus Z$. An additive interval function M (resp., m) defined on $\mathcal{I}$ dS-continuous on K is called a dS-major (resp., dS-minor) function of f if the lower (resp., the upper) d-derivative satisfies the inequality

$$\underline{D}_d M(\mathbf{x}) \geq f(\mathbf{x}) \quad (\text{resp. } \overline{D}_d m(\mathbf{x}) \leq f(\mathbf{x})) \quad \text{for all} \quad \mathbf{x} \in K \setminus Z.$$

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Lemma (monotonicity)

Let an additive interval function R on $\mathcal{I}$ be dS-continuous on K and satisfy the inequality $\underline{D}_d R(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in K \setminus Z$. Then $R(I) \geq 0$ for any interval $I \in \mathcal{I}$.

Perron dyadic Saks continuous integral

Lemma

Let M and m be a dS-major and a dS-minor function for a point-function f on K. Then for each interval $I \in \mathcal{I}$ we have $M(I) \geq m(I)$.

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It implies that for any function f we have $\inf_M\{M(K)\} \geq \sup_m\{m(K)\}$ where "inf" and "sup" are taken over all dS-major and dS-minor function of f, respectively. This justifies the following definition.

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Definition ($P_dS - integral$)

A point-function f defined at least on $K \setminus Z$ is said to be P_dS-integrable on K, if there exists at least one dS-major function and at least one dS-minor function of f and

$$-\infty < \inf_M \{M(K)\} = \sup_m \{m(K)\} < +\infty$$

where "inf" and "sup" are taken as above. The common value is called *P_dS-integral* of f on K and is denoted by $(P_dS) \int_K f$.

Recovering primitive by P_dS-integral

Directly from the definitions we get the following result which shows that the P_dS-integral solves the problem of recovering the primitive from its d-derivative in the form we need.

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Theorem (on recovering primitive)

If an additive dS-continuous interval function F on $\mathcal{I}$ is d-differentiable with $D_dF(\mathbf{x}) = f(\mathbf{x})$ everywhere on $K \setminus Z$ then the function f is P_dS-integrable on K and F is its indefinite P_dS-integral.

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If in the above definition of minor and major functions we assume that inequalities for derivatives are fulfilled everywhere on K and we substitute the assumption of dS-continuity with the d-continuity, we get the definition of P_d integral (which by the way is equivalent to the dyadic Kurzweil-Henstock H_d-integral).

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Theorem (Uniqueness of primitive)

Let an additive interval function R on $\mathcal{I}$ be dS-continuous on K and satisfy the equality $D_d R(\mathbf{x}) = 0 \ \text{ for all } \ \mathbf{x} \in K \setminus Z$. Then $R(I) = 0$ for any interval $I \in \mathcal{I}$.

Recovering primitive by P_dS-integral

In the above theorem the stronger assumption of dS-continuity of F is essential. In fact if we suppose only d-continuity for the additive function F we will lose the uniqueness of primitive for set function. This example of function F defined in $K = [0, 1] \times [0, 1]$ can show it:

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$$F(A) = \begin{cases} 1 & if \quad A = K \\ 0 & if \quad A = I \; with \; I \cap (\{0\} \times [0,1]) = \emptyset \\ \frac{1}{2^n} & if \quad A = I_0^0 \times I_j^{(n)} \end{cases}$$

and using the additivity we extend the definition of this function F on any $I \in \mathcal{I}$.

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This function, not being trivial, has derivative equal zero everywhere in $K \setminus \{0\} \times [0,1]$.

P_dS-integral

The next theorem shows in particular that the P_d-integral, constructed for function defined on K using d-continuous functions instead of dS-continuous, (and also the equivalent H_d-integral) fails to solve the problem of recovering the primitive under assumption of the above theorem (with the exceptional set Z).

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Theorem

There exists a dS-continuous function Φ which is d-differentiable everywhere on $[0,1]^2 \setminus (\{0\} \times [0,1])$ but its d-derivative $D_d(\Phi)$ being P_dS-integrable is not P_d-integrable.

We mention although H_d-integral is dS-continuous we do not know whether it can be defined by Perron method using dS-continuous major and minor functions. In this connection we are leaving open the following problem:

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Open Problem

Is any P_d-integrable function P_dS-integrable?

Main results

Applying the equality (5), then "cross" lemma related to the dS-continuity of quasi measure, theorem on recovering the primitive and the proposition giving necessary and sufficient condition for series to be Fourier series, we can formulate the main theorem for Walsh series:

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Theorem (recovering coefficients for Walsh series)

If a two-dimensional Walsh series is rectangular convergent to a sum f everywhere in $K \setminus Z$ then f is P_dS-integrable on K and the coefficients of the series are P_dS-Fourier coefficients of f.

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Using corresponding propositions and lemma for Haar series we get

Theorem (recovering coefficients for Haar series)

If a two-dimensional Haar series is rectangular convergent to a sum f everywhere in K then f is P_dS-integrable on K and the coefficients of the series are P_dS-Fourier coefficients of f.

Remark on uniqueness of solution

In the multidimensional case the solution of the uniqueness of coefficients depends on the type of convergence. In Haar series with ρ-regular convergence the theorem of uniqueness is not always true. Let us mention an interesting result obtained by Plotnikov who proved by constructing an example that for ρ-regular convergence with $\rho \in (\frac{\sqrt{2}}{2}, 1]$ (in particular for so called cubic convergence) the theorem of uniqueness does not hold. If the uniqueness is guarantied then the complete solution of the problem of recovering the coefficients of a summable series with respect to some system means that a process of integration is developed so that any such a series is the Fourier series of its sum, in the sense of this integral. More complicated types of continuity and corresponding types of Perron integral were introduced by Plotnikov to deal with the ρ-regular convergent series with ρ close to zero.

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THANK YOU FOR YOUR ATTENTION!!