Symmetrization and convexity I

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(joint work with A. Witkowski)

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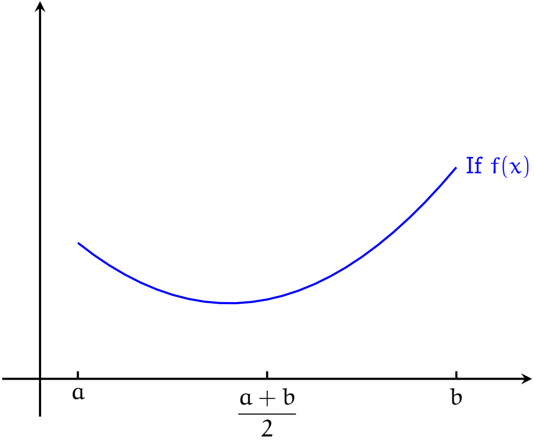
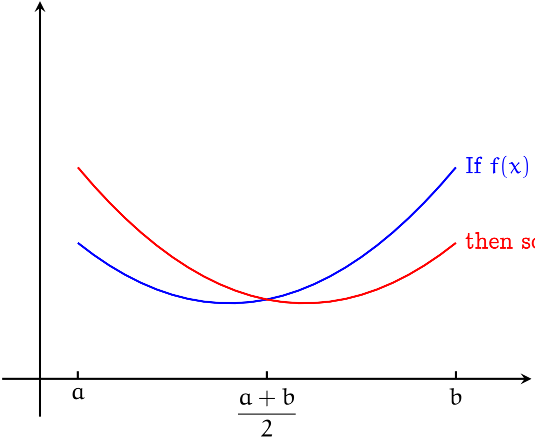
Symmetrization – 1 variable



If $f(x)$ is convex on $[a, b]$,

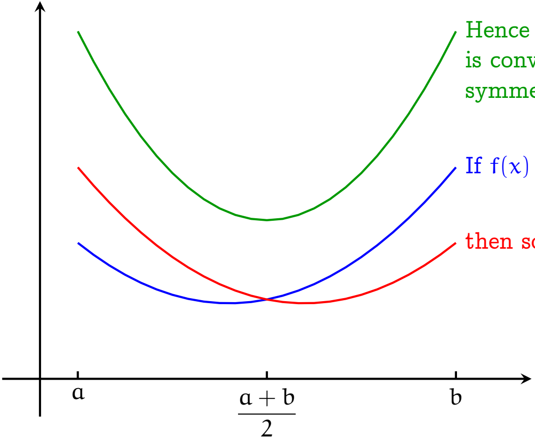
Symmetrization – 1 variable



If $f(x)$ is convex on $[a, b]$,

then so is $f(a + b - x)$.

Symmetrization – 1 variable



Hence $f(x) + f(a + b - x)$ is convex on $[a, b]$ and symmetric wrt. the midpoint

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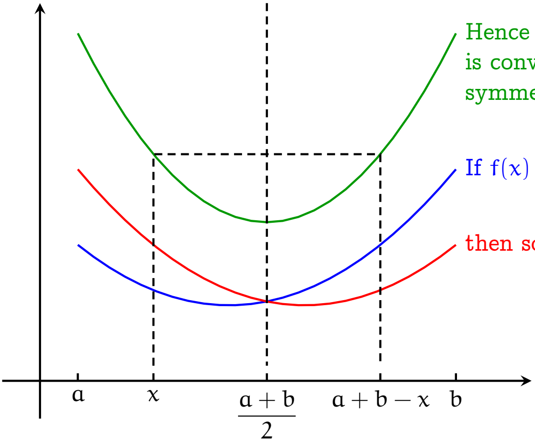

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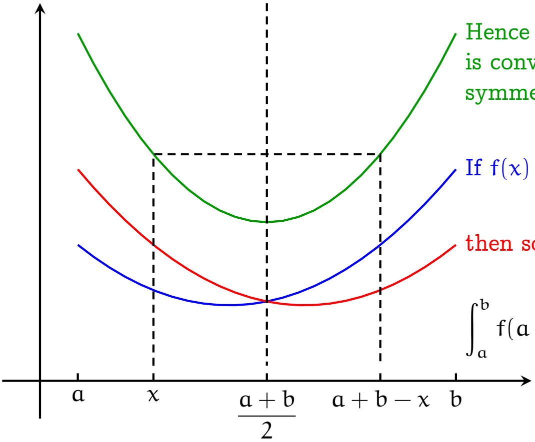

Hence $f(x) + f(a+b-x)$ is convex on $[a,b]$ and symmetric wrt. the midpoint

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$$\int_a^b f(a+b-x)\,dx = \int_a^b f(x)\,dx$$

Symmetrization – 1 variable

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Properties

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Symmetrization – 1 variable

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Symmetrization – 1 variable

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Symmetrization of a convex function

Suppose f is convex on $[a, b]$.

Symmetrization – 1 variable

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Symmetrization – 1 variable

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Symmetrization of a convex function

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- g is convex
- $g\left(\frac{a+b}{2}\right) = 2f\left(\frac{a+b}{2}\right) \leqslant f(x) + f(a+b-x) = g(x) \leqslant g(a)$

Symmetrization – 1 variable

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Hermite–Hadamard inequality by symmetrization

Suppose f is convex on $[a, b]$.

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Hermite–Hadamard inequality by symmetrization

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Symmetrization – 1 variable

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- $2(b-a)f\left(\frac{a+b}{2}\right) \leqslant \int_a^b \big(f(x) + f(a+b-x)\big)\,dx$
  $\leqslant (b-a)\big(f(a) + f(b)\big)$

Symmetrization – 1 variable

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- $2(b-a)f\left(\frac{a+b}{2}\right) \leqslant \int_a^b \big(f(x)+f(a+b-x)\big)\,dx$
  $\leqslant (b-a)\big(f(a)+f(b)\big)$
- $2(b-a)f\left(\frac{a+b}{2}\right) \leqslant 2\int_a^b f(x)\,dx \leqslant (b-a)\big(f(a)+f(b)\big)$

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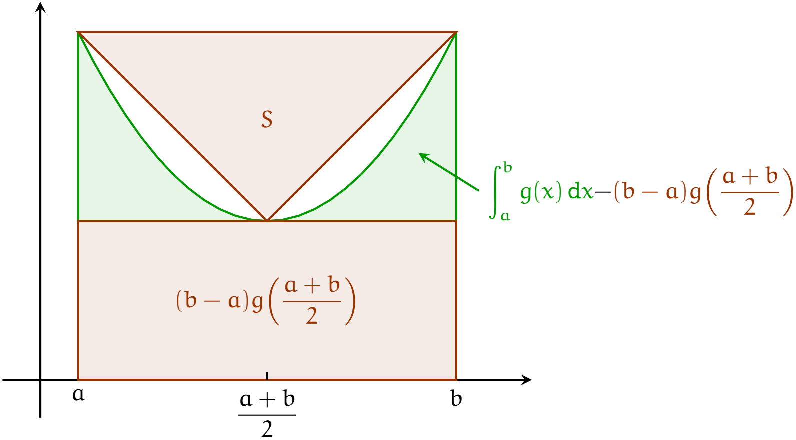
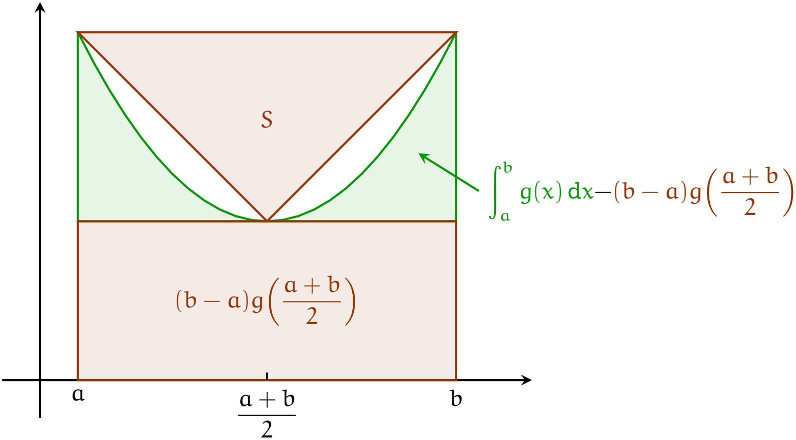
  $\leqslant (b-a)\big(f(a) + f(b)\big)$
- $2(b-a)f\left(\frac{a+b}{2}\right) \leqslant 2\int_a^b f(x)\,dx \leqslant (b-a)\big(f(a) + f(b)\big)$
- $f\left(\dfrac{a+b}{2}\right) \leqslant \dfrac{1}{b-a}\int_a^b f(x)\,dx \leqslant \dfrac{f(a)+f(b)}{2}$

Symmetrization – 1 variable



$$\int_a^b g(x)\,dx - (b-a)g\left(\frac{a+b}{2}\right) \leqslant S$$

Symmetrization – 1 variable



$$\int_a^b g(x)\,dx-(b-a)g\left(\frac{a+b}{2}\right) \leqslant S \leqslant (b-a)g(a)-\int_a^b g(x)\,dx$$

LHH estimates the integral MV better than RHH

Suppose f is convex on $[a, b]$.

Symmetrization – 1 variable

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LHH estimates the integral MV better than RHH

Suppose f is convex on $[a, b]$.

- $f\left(\dfrac{a+b}{2}\right) \leqslant \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx \leqslant \dfrac{f(a)+f(b)}{2}$ HH inequality

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- $0 \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)$ LHH

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Symmetrization – 1 variable

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- $0 \leqslant \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx$ RHH
- f convex on $[a,b] \implies 0 \leqslant \mathrm{LHH} \leqslant \mathrm{RHH}$

Symmetrization – 1 variable

Proof by symmetrization

Symmetrization – 1 variable

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Proof by symmetrization

- $0 \leqslant \int_a^b g(x)\,dx - (b-a)g\Big(\frac{a+b}{2}\Big) \leqslant (b-a)g(a) - \int_a^b g(x)\,dx$

Symmetrization – 1 variable

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- $0 \leqslant \int_a^b g(x)\,dx - (b-a)g\left(\frac{a+b}{2}\right) \leqslant (b-a)g(a) - \int_a^b g(x)\,dx$
- $0 \leqslant 2\int_a^b f(x)\,dx - 2(b-a)f\left(\frac{a+b}{2}\right)$
  $$\leqslant (b-a)\big(f(a)+f(b)\big) - 2\int_a^b f(x)\,dx$$

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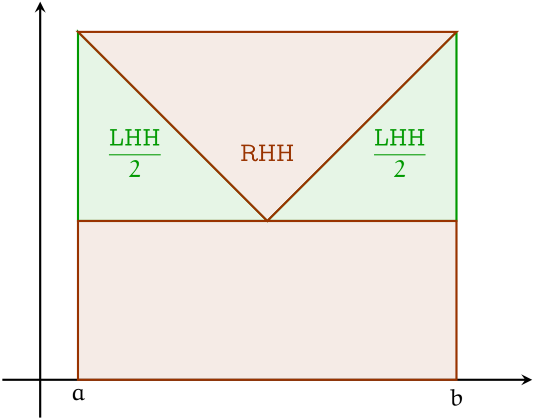
Proof by symmetrization

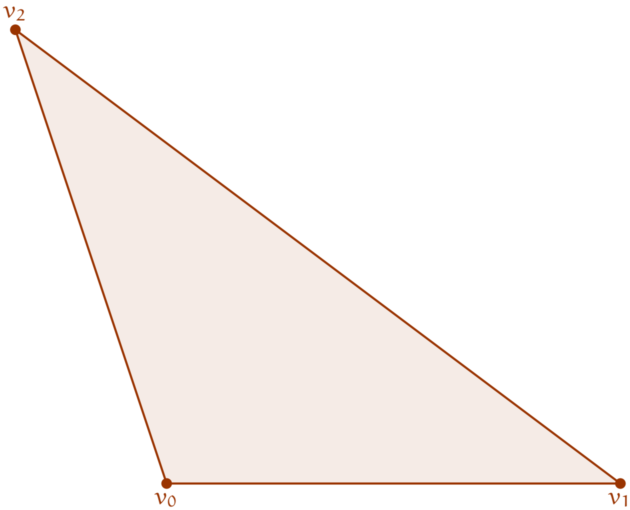
- $$0 \leqslant \int_a^b g(x)\,dx - (b-a)g\Big(\frac{a+b}{2}\Big) \leqslant (b-a)g(a) - \int_a^b g(x)\,dx$$
- $$0 \leqslant 2\int_a^b f(x)\,dx - 2(b-a)f\Big(\frac{a+b}{2}\Big) \leqslant (b-a)\big(f(a)+f(b)\big) - 2\int_a^b f(x)\,dx$$
- Divide both sides by $2(b-a)$.

The inequality $0 \leqslant \mathrm{LHH} \leqslant \mathrm{RHH}$ is optimal



$\mathrm{LHH} = \mathrm{RHH}$

Symmetrization – 2 variables



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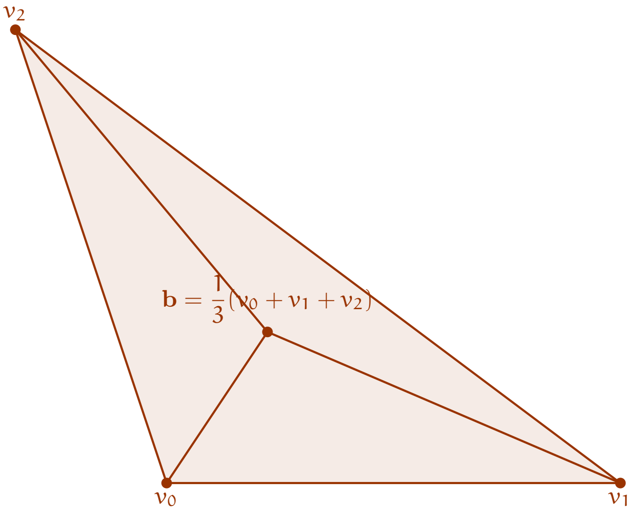

Symmetrization – 2 variables

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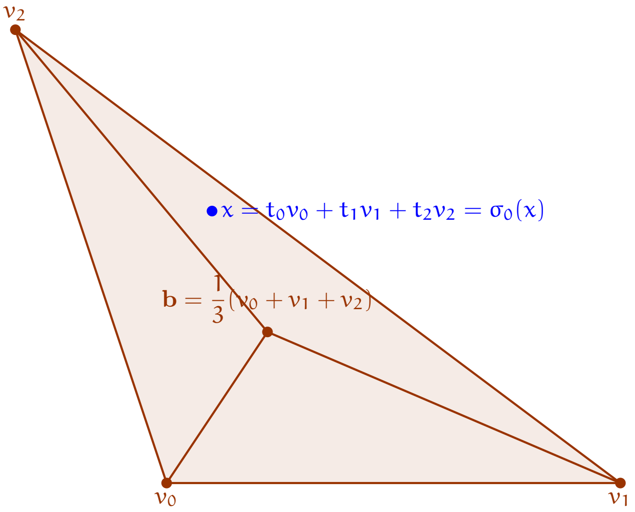

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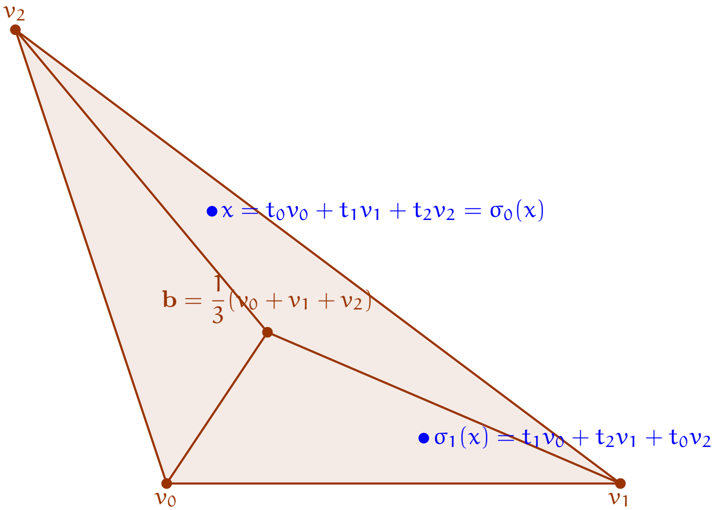

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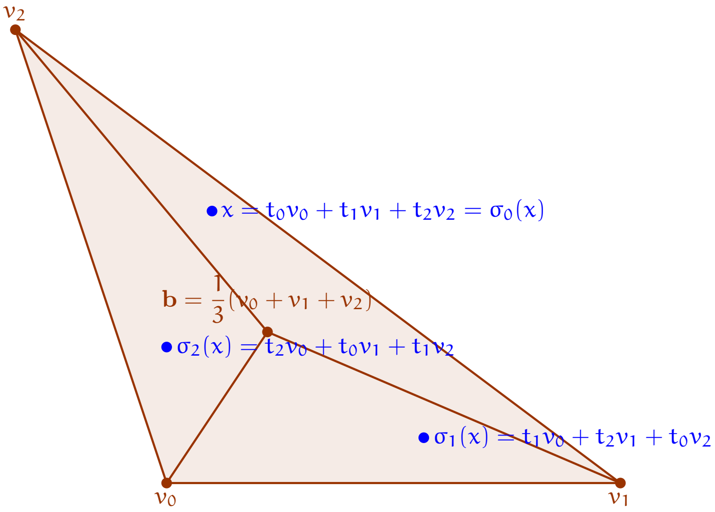
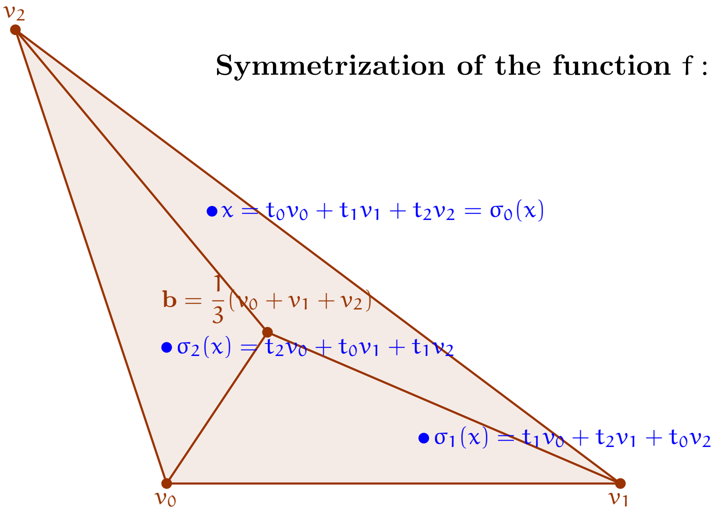
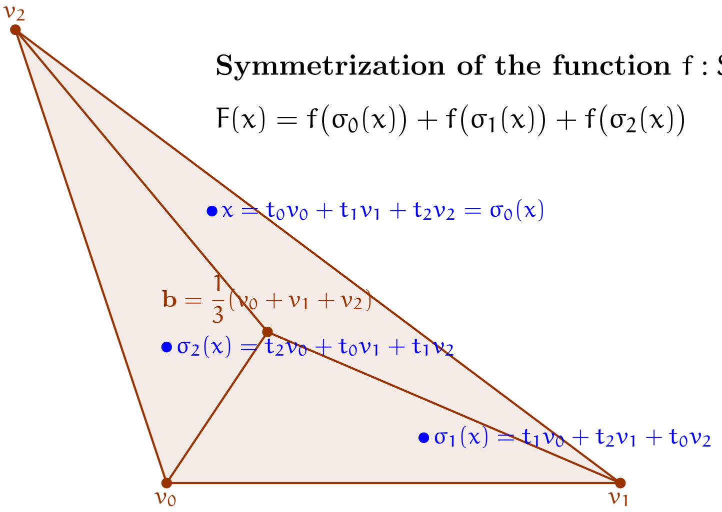


Symmetrization of the function $f : S \to \mathbb{R}$



Symmetrization of the function $f : S \to \mathbb{R}$

$$F(x) = f\big(\sigma_0(x)\big) + f\big(\sigma_1(x)\big) + f\big(\sigma_2(x)\big)$$



Symmetrization – 2 variables

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Symmetrization – 2 variables

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Symmetrization – 2 variables

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- $\displaystyle\int_S F(x)\,dx = 3\int_S f(x)\,dx$
- f is convex on $S \implies F$ is convex on S

Symmetrization – 2 variables

Hermite–Hadamard inequality

$S \subset \mathbb{R}^2$ – a simplex (a triangle) with vertices v_0, v_1, v_2 and barycenter $\mathbf{b}$, $f : S \to \mathbb{R}$ – a convex function

$$f(\mathbf{b}) \leqslant \frac{1}{\operatorname{vol}(S)} \int_S f(x)\,dx \leqslant \frac{f(v_0) + f(v_1) + f(v_2)}{3}$$

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The inequality between LHH and RHH terms

Symmetrization – 2 variables

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The inequality between LHH and RHH terms

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Symmetrization – 2 variables

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Symmetrization – 2 variables

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The inequality between LHH and RHH terms

- $\displaystyle \frac{1}{\operatorname{vol}(S)} \int_S f(x)\,dx - f(\mathbf{b})$ LHH
- $\displaystyle \frac{f(v_0) + f(v_1) + f(v_2)}{3} - \frac{1}{\operatorname{vol}(S)} \int_S f(x)\,dx$ RHH
- $f : S \to \mathbb{R}$ convex $\implies 0 \leqslant \mathrm{LHH} \leqslant 2\,\mathrm{RHH}$

Symmetrization – 2 variables

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The proof goes by symmetrization F.

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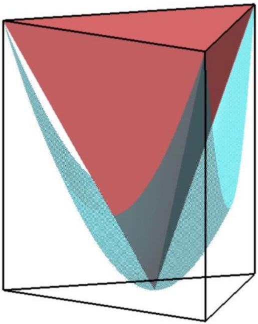

Symmetrization – 2 variables

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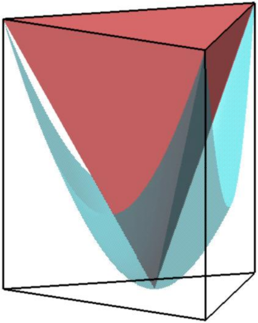

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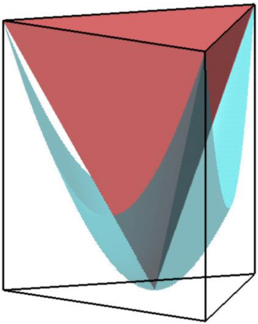

- Volume between the base and the surface

$$0 \leqslant \int_S F(x)\,dx - F(\mathbf{b}) \cdot \mathrm{area(base)}$$
$$= \mathrm{area(base)} \cdot \mathrm{LHH}(F)$$

Symmetrization – 2 variables

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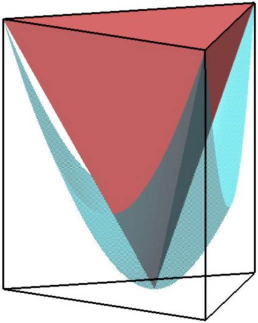

- Volume between the base and the surface

$$0 \leqslant \int_S F(x)\,dx - F(\mathbf{b}) \cdot \operatorname{area}(\text{base}) = \operatorname{area}(\text{base}) \cdot \operatorname{LHH}(F)$$

- $\operatorname{area}(\text{base}) \cdot \operatorname{LHH}(F) \leqslant 2\,\operatorname{vol}(\text{pyramid})$

Symmetrization – 2 variables

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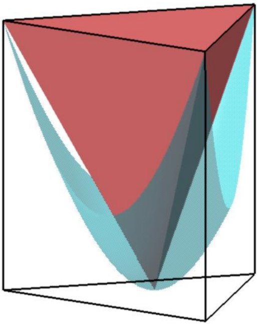

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$$= \operatorname{area}(\text{base}) \cdot \operatorname{LHH}(F)$$

- $\operatorname{area}(\text{base}) \cdot \operatorname{LHH}(F) \leqslant 2\operatorname{vol}(\text{pyramid})$
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Symmetrization – 2 variables

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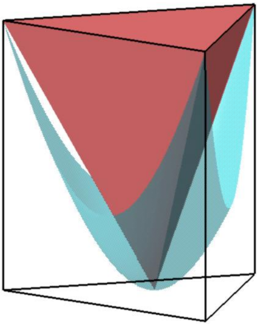
The proof goes by symmetrization F.



- Volume between the base and the surface

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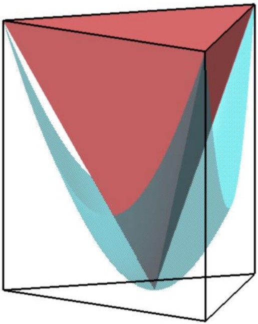

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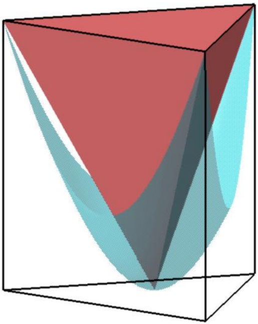

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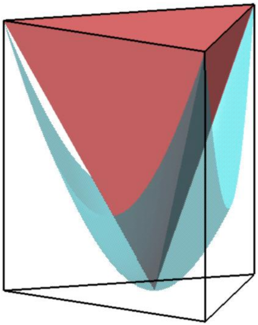

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Symmetrization – 2 variables

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- That is why the inequality in question is optimal.

Symmetrization – n variables

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- We identify $\sigma \in C$ with the affine transformation $\sigma : S \to S$ given by

$$\sigma\Big(\sum_{i=0}^{n} t_i v_i\Big) = \sum_{i=0}^{n} t_{\sigma(i)} v_i$$

Return for a while to $n = 2$.

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Symmetrization of a convex function

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Hermite–Hadamard inequality

$S \subset \mathbb{R}^n$ – a simplex with vertices $v_0, \ldots, v_n$ and barycenter $\mathbf{b}$,
$f : S \to \mathbb{R}$ – a convex function

$$f(\mathbf{b}) \leqslant \frac{1}{\operatorname{vol}(S)} \int_S f(x)\, dx \leqslant \frac{f(v_0) + \cdots + f(v_n)}{n+1}$$

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- $\displaystyle \frac{f(v_0) + \cdots + f(v_n)}{n+1} - \frac{1}{\operatorname{vol}(S)} \int_S f(x)\,dx$ RHH
- $f : S \to \mathbb{R}$ convex $\implies 0 \leqslant \mathrm{LHH} \leqslant n\,\mathrm{RHH}$

To be continued

A. Witkowski, Sz. Wąsowicz, *On some inequality of Hermite–Hadamard type*, Opuscula Math. 32 (2012), 591–600.

Thank you very much for your kind attention