Cantor's intersection theorem for cone metric spaces and its applications to fixed point theory

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Cone metric spaces

Definition.

Let E be a real Banach space and $K \subset E$. K is called a cone iff:

(*i*) K is closed, nonempty and $K \neq \{0\}$;

(*ii*) if $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in K$, then $ax + by \in K$;

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A cone metric space is a pair (X, d), where X is a nonempty set and $d : X \times X \to E$ satisfies three well-known axioms of a metric with respect to the following partial ordering $\leq$ in E: $x \leq y$ iff $y - x \in K$.

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Lemma.

Let $E = \mathbb{R}^n$ and $K = \{x \in \mathbb{R}^n : x_i \geq 0\}$. A function $d : X \times X \to \mathbb{R}^n$ $(d = (d_1, ..., d_n))$ is a cone metric iff $\{d_1, ..., d_n\}$ is a separating family of pseudometrics.

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We say that a sequence (x_n) of elements of X is convergent to $x \in X$ if for any $0 \ll c$, there exists $k \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n \geq k$, and x is called the limit of (x_n). Similarly, we define Cauchy sequence. Then the completeness of (X, d) is understood in an analogous way as in the case of real-valued metric.

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It is easy to show that $B(x_0, r)$ is a closed set.

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Let B be a subset of X. The diameter of B is the vector $\delta(B) := \sup \{d(x, y) : x, y \in B\}$ (if it exists).

Cantor's intersection theorem

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Theorem (S. H. Alnafei, S. Radenović, N. Shahzad; 2011).

A cone metric space (X, d) over a strongly minihedral cone K is complete iff every decreasing sequence (A_n) of nonempty closed sets with $\delta(A_n) \xrightarrow{\|\cdot\|} 0$, has nonempty intersection.

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Theorem.

Let (X, d) be a cone metric space. The following statements are equivalent:

(*i*) X is complete;

(*ii*) every decreasing sequence (A_n) of nonempty closed sets such that every A_n has a bound $c_n \in K$ and $c_n \xrightarrow{\|\cdot\|} 0$, has a nonempty intersection;

(*iii*) every decreasing sequence $((B(x_n, r_n))$ of closed balls with $r_n \xrightarrow{\|\cdot\|} 0$, has a nonempty intersection.

Banach's fixed point theorem

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Theorem.

Let K be a cone in a Banach space E and $\Lambda : E \to E$ be a linear bounded operator, which is monotone, i.e., $a \le b$ implies $\Lambda a \le \Lambda b$ for $a, b \in K$. Let (X, d) be a complete cone metric space and $T : X \to X$ be such that

$$d(Tx, Ty) \le \Lambda(d(x, y)) \textit{ for all } x, y \in X.$$

If $\lim\limits_{n\to\infty} \|\Lambda^n\|^{\frac{1}{n}} < 1$, then T has a unique fixed point x_* and for any $x_0 \in X$, $x_* = \lim\limits_{n\to\infty} T^n x_0$.

Remarks

Remark 1.

We can prove Banach's fixed point theorem using Cantor's intersection theorem in a similar way as in real-valued metric spaces:

(i) We consider sets $A_n := \left\{x \in X : d(x, Tx) \leq \frac{1}{n}c\right\}$.

(ii) The sequence (A_n) satisfies all the asumptions of Cantor's intersection theorem (the proof that the sets A_n are closed is different from the one in real-valued metric case).

(iii) There exists unique $x_* \in \bigcap\limits_{n\in\mathbb{N}} A_n$ and $Tx_* = x_*$.

Remark 2.

It can happen that $x_n \longrightarrow x$ and $y_n \longrightarrow y$ but $d(x_n, y_n) \overset{\|\cdot\|}{\not\longrightarrow} d(x, y)$, so a cone metric d need not be a continuous function.

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Example. Let $E = C^{(1)}[0,1]$ with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$, $K = \{x \in E : x(t) \geq 0, \textit{for all } t \in [0,1]\}$. We define a cone metric $d : K \times K \to K$,

$$d(x,y) := \begin{cases} 0, & x = y \\ x + y, & x \neq y \end{cases} .$$

Consider $x_n(t) := \frac{1+sin(nt)}{n}, x(t) = y_n(t) = y(t) = 0$, then $x_n \longrightarrow x, y_n \longrightarrow y$, but $d(x_n, y_n) \overset{\|\cdot\|}{\not\longrightarrow} d(x, y)$.

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