Resolvability properties of similar topologies

Sebastian Lindner

University of Łódź

Similarity

Definition

Let $\mathcal{F}, \mathcal{G} \subset 2^X \setminus \{\emptyset\}$. We say, that $\mathcal{F}$ is coinitial to $\mathcal{G}$ if for every $G \in \mathcal{G}$ there exists $F \in \mathcal{F}$ such that $F \subset G$.

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Let us consider a non-empty space X with two topologies, $\mathcal{T}_1$ and $\mathcal{T}_2$. We will say that $\mathcal{T}_1$ and $\mathcal{T}_2$ are similar ($\mathcal{T}_2 \simeq_s \mathcal{T}_1$) iff for every $A \subset X$ we have

$$\mathrm{Int}_{\mathcal{T}_1}(A) \neq \emptyset \iff \mathrm{Int}_{\mathcal{T}_2}(A) \neq \emptyset.$$

Similarity

The relation of similarity was investigated among others in [BFKT]. In that note one can find the following useful characterization:

Theorem, [Bartoszewicz, Filipczak, Kowalski, Terepeta 2014]

The following statements are equivalent

- $\mathcal{T}_1 \simeq_s \mathcal{T}_2$
- $\mathcal{D}(X, \mathcal{T}_1) = \mathcal{D}(X, \mathcal{T}_2)$
- $(\mathcal{NB}(X, \mathcal{T}_1), \mathcal{ND}(X, \mathcal{T}_1)) = (\mathcal{NB}(X, \mathcal{T}_2), \mathcal{ND}(X, \mathcal{T}_2))$.
- $\mathcal{T}_1 \setminus \{\emptyset\}$ *and* $\mathcal{T}_2 \setminus \{\emptyset\}$ *are mutually coinitial.*

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- $\mathcal{T}_1 \setminus \{\emptyset\}$ *and* $\mathcal{T}_2 \setminus \{\emptyset\}$ *are mutually coinitial.*

Moreover, for every topological space $(X, \mathcal{T})$ the family $\mathcal{NB}(X, \mathcal{T})$ forms an algebra of sets, and $\mathcal{ND}(X, \mathcal{T})$ is an ideal contained in $\mathcal{NB}(X, \mathcal{T})$.

Similarity

Let Φ be an arbitrary property of topological spaces.

Definition

We will say that Φ is $\simeq_s$-proof iff for every topology $\mathcal{T}_1$ having the property Φ every similar topology $\mathcal{T}_2$ also has the property Φ.

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For any cardinal κ we say that $(X, \mathcal{T})$ is κ-resolvable iff there exists a family of cardinality κ of pairwise disjoint dense subsets of X.

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The dense-in-itself space is called maximally resolvable if it is $\Delta(X, \mathcal{T})$-resolvable, where $\Delta(X, \mathcal{T}) = \min\{|G| \ : \ G \in \mathcal{T}, G \neq \emptyset\}$.

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The space $(X, \mathcal{T})$ is called extra-resolvable, if there exists the family $\mathbf{D}$ of dense subsets of X such that $|\mathbf{D}| > \Delta(X, \mathcal{T})$ and for every $A, B \in \mathbf{D}$ we have $A = B$ or $A \cap B \in \mathcal{ND}(X, \mathcal{T})$.

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Theorem, [Rose, Sizemore, Thurston 2006]

$submaximal \implies HI \implies OHI$

Observation

The following properties and parameters are $\simeq_s$-proof:

- κ-resolvability
- $\Delta(X, \mathcal{T})$
- maximal resolvability
- extra resolvability
- being dense-in-itself

Hewitt's decomposition

Theorem, [Hewitt 1943]

Every topological space $(X, \mathcal{T})$ can be uniquely represented as a disjoint union

$$X = F_X \cup G_X,$$

where F_X is closed and resolvable and G_X is open and hereditary irresolvable. This pair of sets is called the Hewitt decomposition of the space X.

Results

Definition

Let $\mathcal{A}$ be an arbitrary algebra. By $\mathcal{H}(\mathcal{A})$ we shall denote the maximal hereditary subfamily of $\mathcal{A}$:

$$\mathcal{H}(\mathcal{A}) = \{A \in \mathcal{A} : \forall_{B \subset A} B \in \mathcal{A}\}.$$

Theorem

Let $(X, \mathcal{T})$ - an arbitrary topological space. Let (F_X, G_X) form the Hewitt decomposition of the space X. Then

$$\mathcal{H}(\mathcal{NB}) = \{A \cup N \ : \ A \subset G_X \ \wedge \ N \in \mathcal{ND}\}.$$

Results

Corollary

$(X,\mathcal{T})$ is resolvable iff $\mathcal{H}(\mathcal{NB})=\mathcal{ND}$.

Half of this equivalence can be found in [BFKT].

OHI

Theorem

The following statements are equivalent:

- $(X, \mathcal{T})$ *is OHI*
- $(\mathcal{H}(\mathcal{NB}) \setminus \mathcal{ND})$ *is coinitial to* $(\mathcal{NB} \setminus \mathcal{ND})$.
- $\mathcal{NB} = 2^X$

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Corollary

OHI is $\simeq_s$*-proof.*

Submaximality and HI

Theorem, [Crossley 1974]

For every topological space $(X, \mathcal{T})$ the family

$$\mathcal{F}(\mathcal{T}) = \{G \setminus N : G \in \mathcal{T}, N \in \mathcal{ND}\}$$

forms a topology, similar to $\mathcal{T}$ and NODEC.

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HI and submaximality are not $\simeq_s$-proof.

Similarity is not hereditary

Corollary

The similarity $(X, \mathcal{T}_1) \simeq_s (X, \mathcal{T}_2)$ *does not necessarily imply the relation* $(A, \mathcal{T}_1|_A) \simeq_s (A, \mathcal{T}_2|_A)$.

Theorem

Let $(X, \mathcal{T}_1) \simeq_s (X, \mathcal{T}_2)$.
If $A \in \mathcal{SO}(\mathcal{T}_1) \cap \mathcal{SO}(\mathcal{T}_2)$ *then* $(A, \mathcal{T}_1|_A) \simeq_s (A, \mathcal{T}_2|_A)$.

When the pair $(\mathcal{A}, \mathcal{I})$ is topological?

Let $\mathcal{A} \subset 2^X$ be an arbitrary algebra and $\mathcal{I} \subset \mathcal{A}$ - an arbitrary ideal of sets.

Definition

We say that the pair $(\mathcal{A}, \mathcal{I})$ is:

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- LDO if there exists lower density operator Φ on $(\mathcal{A}, \mathcal{I})$.

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$$\forall_{\mathcal{B} \subset \mathcal{A}} \quad \bigcup_{B \in \mathcal{B}} (B \cap \Phi(B)) \in \mathcal{A}.$$

- TOP if there exists a topology $\mathcal{T}$ on X such that $(\mathcal{A}, \mathcal{I}) = (\mathcal{NB}, \mathcal{ND})$.

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Theorem, [Hejduk]

$$LDO \wedge HULL \implies TOP \implies HULL$$

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Theorem

$TOP \iff LDO+$

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Corollary

Let $\mathcal{I} \subset 2^X$ is an arbitrary ideal. The following statements are eqivalent:

- $(2^X, \mathcal{I})$ *is TOP.*
- *there exists an submaximal topology $\mathcal{T}$ on X such that* $(\mathcal{A}, \mathcal{I}) = (\mathcal{NB}, \mathcal{ND})$.
- $(2^X, \mathcal{I})$ *is LDO.*

When the pair $(\mathcal{A}, \mathcal{I})$ is topological?

Corollary

Let $(\mathcal{A}, \mathcal{I})$ *be such that* $(\mathcal{H}(\mathcal{A}) \setminus \mathcal{I})$ *is coinitial to* $(\mathcal{A} \setminus \mathcal{I})$. *Then the following holds*

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Theorem, [Zindulka]

Every analytic set $B \subset \mathbb{R}$ contains a universal measure zero set E such that $dim_H E = dim_H B$.

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Let $\mathcal{L}$ - σ-ideal of Lebesgue measurable sets. Then $\mathcal{H}(\mathcal{L}) = \mathcal{N}$. For $\alpha \in [0, 1)$ let $\mathcal{I}_\alpha = \{E \in \mathcal{L} \ : \ \dim_H(E) \leq \alpha\}$.

Corollary

For $\alpha \in [0, 1)$ the pair $(\mathcal{L}, \mathcal{I}_\alpha)$ is not TOP.

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