Extending generalized probability measures II

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Abstract

Various systems of [0,1]-valued functions model generalized random events and generalized probability measures. A problem related to the extension of generalized probability measures (the existence of certain epireflection) given in

J. Havlíčková: Real functions and the extension of generalized probability measures. Tatra Mt. Math. Publ. **55** (2013),85–94.

has been recently solved by R. Frič. We show that the solution leads to a better understanding of IF-probability (developed by B. Riečan) and its relationship to fuzzy probability.

Preliminaries

Denote I the closed unit interval [0,1]. Let X be a set. Then fuzzy subsets on X are maps of X into I. We identify a subset $A \subseteq X$ and its indicator function χ_A (where $\chi_A(x) = 1$ for $x \in A$ and $\chi_A(x) = 0$ otherwise). If $\mathbb{A}$ is a σ-algebra of subsets of X, then $\mathcal{M}(\mathbb{A})$ will denote the $\mathbb{A}$-measurable fuzzy subsets of X.

Recall that

ID-poset ... a system $\mathcal{X} \subseteq I^X$ of fuzzy subsets of X carrying a D-poset structure: $\mathcal{X}$ is partially ordered (coordinatewise order), $0_X, 1_X \in \mathcal{X}$, if $u, v \in \mathcal{X}$ and $v \leq u$, then $u - v \in \mathcal{X}$; $\mathcal{X}$ is called a D-poset of fuzzy sets

ID ... the category of *ID*-posets and sequentially continuous (with respect to the coordinatewise convergence of sequences) maps preserving the D-poset structure

BID and *CGBID*

bold algebra ... a system $\mathcal{X} \subseteq I^X$ of fuzzy subsets of X such that $0_X, 1_X \in \mathcal{X}$ and $\mathcal{X}$ is closed with respect to the Łukasiewicz operations $\oplus$, $\odot$, "complement": if $u, v \in \mathcal{X}$, then
$(u \oplus v)(x) = min\{u(x) + v(x), 1\}$,
$(u \odot v)(x) = max\{u(x) + v(x) - 1, 0\}$,
$u^c(x) = 1 - u(x), x \in \mathcal{X}$

Bold algebras generalize Boolean algebras.

BID ... the category of bold algebras as objects and sequentially continuous D-homomorphisms as morphisms

CGBID ... the full subcategory of *BID*, the objects are bold algebras of the form $\mathcal{M}(\mathbb{A})$; if X is a one-point set $\{a\}$, then $\mathbb{A}$ is a trivial σ-algebra $\mathbb{T} = \{\emptyset, \{a\}\}$ and $\mathcal{M}(\mathbb{T}) = I^{\{a\}} = [0, 1]$

Morphisms

Let $\mathbb{A}$ be a σ-algebra of subsets of X and let p be a probability measure on $\mathbb{A}$. For $f \in \mathcal{M}(\mathbb{A})$ put $h(f) = \int f \, dp$ and denote h the resulting map. Recall that
(i) h is a morphism (i.e. a sequentially continuous map preserving the D-poset structure).
(ii) If $h : \mathcal{M}(\mathbb{A}) \longrightarrow [0,1]$ is a morphism, then there is a unique probability measure p on $\mathbb{A}$ such that $h(f) = \int f \, dp$, $f \in \mathcal{M}(\mathbb{A})$.

fuzzy probability ... fuzzy random events are of the form $\mathcal{M}(\mathbb{A})$ and fuzzy probability measures are integrals

From *BID* to *CGBID*

Let $\mathcal{X}_0 \subseteq I^X$ be a bold algebra and let $\mathcal{X}$ be the smallest sequentially closed subset of I^X containing $\mathcal{X}_0$ ($\mathcal{X}$ is a bold algebra). Then there is a unique σ-algebra $\mathbb{A}_{\mathcal{X}}$ of subsets of X such that $\mathbb{A}_{\mathcal{X}} \subseteq \mathcal{X} \subseteq \mathcal{M}(\mathbb{A}_{\mathcal{X}})$. If, moreover, $\mathcal{X}$ contains all constant functions $c_X, c \in [0,1]$, then $\mathcal{X} = \mathcal{M}(\mathbb{A}_{\mathcal{X}})$.

Recently has been proved that *CGBIG* is an epireflective subcategory of *BID* ($\mathcal{M}(\mathbb{A}_{\mathcal{X}})$ is the epireflection of $\mathcal{X}_0$).

R. Frič: On *D*-posets of fuzzy sets. Math.Slovaca **64** (2014), 545–554.

This solves a problem posed in

J. Havlíčková: Real functions and the extension of generalized probability measures. Tatra Mt. Math. Publ. **55** (2013),85–94.

IF probability

IF set ... a pair (u, v), where u, v are fuzzy subsets of X such that $u(x) + v(x) \leq 1$ for all $x \in X$

IF sets have been intrudoced by K. T. Atanassov as a generalization of fuzzy sets. Probability on *IF* events has been developed by B. Riečan. *IF* events (suitable pairs of fuzzy sets) can be modeled via powers of a bold algebra. Basic notions of the *IF* probability have been outlined in the preceding talk.

Products

Let $\mathcal{X} \subseteq I^X$ and $\mathcal{Y} \subseteq I^Y$ be D-posets of fuzzy sets, i.e., objects of ID. Let $\mathcal{Z} \subseteq I^Z$ be their product in ID. Then $\mathcal{Z}$ consists of all pairs $(u, v), u \in \mathcal{X}, v \in \mathcal{Y}$, where the ID-structure (partial order, difference, convergence) is defined coordinatewise, and Z is the disjoint union of X and Y (their coproduct in the category of sets and maps). Each $w = (u, v)$ can be visualized as a function on Z, where u and v are "disjoinly glued" to form w.

If $\mathcal{Y}$ is a bold algebra, then $\mathcal{Y} \times \mathcal{Y}$ denotes the corresponding power bold algebra.

Powers

Let $\mathcal{X}_0 \subseteq I^X$ be a bold algebra, let $\mathcal{X} \subseteq I^X$ be the the smallest sequentially closed bold algebra containing $\mathcal{X}_0$, and let $\mathbb{A}_{\mathcal{X}}$ be the unique σ-algebra of subsets of X such that $\mathbb{A}_{\mathcal{X}} \subseteq \mathcal{X} \subseteq \mathcal{M}(\mathbb{A}_{\mathcal{X}})$. Pairs $(u, v) \in \mathcal{M}(\mathbb{A}_{\mathcal{X}}) \times \mathcal{M}(\mathbb{A}_{\mathcal{X}})$ are fuzzy random events "related to" *IF* random events. The power bold algebra $\mathcal{M}(\mathbb{A}_{\mathcal{X}}) \times \mathcal{M}(\mathbb{A}_{\mathcal{X}})$ carries the usual coordinatevise Łukasiewicz operations: $\oplus, \odot$ and complementation. Denote

$$A(\mathcal{X}_0 \times \mathcal{X}_0) = \{(u, v) \in \mathcal{X}_0 \times \mathcal{X}_0;\ u \leq v\}$$

and

$$A(\mathcal{M}(\mathbb{A}_{\mathcal{X}}) \times \mathcal{M}(\mathbb{A}_{\mathcal{X}})) = \{(u, v) \in \mathcal{M}(\mathbb{A}_{\mathcal{X}}) \times \mathcal{M}(\mathbb{A}_{\mathcal{X}});\ u \leq v\}.$$

Clearly, both $A(\mathcal{X}_0 \times \mathcal{X}_0)$ and $A(\mathcal{M}(\mathbb{A}_{\mathcal{X}}) \times \mathcal{M}(\mathbb{A}_{\mathcal{X}}))$ are closed with respect to the Łukasiewicz operations $\oplus$ and $\odot$, but are not closed with respect to the complementation.

Complementation

THEOREM 1. Let $\mathcal{X}_0 \subseteq I^X$ be a bold algebra. Then $\mathcal{X}_0 \times \mathcal{X}_0$ is the smallest bold algebra containing $A(\mathcal{X}_0 \times \mathcal{X}_0)$.

Proof. Assume that $(u, v) \in \mathcal{X}_0 \times \mathcal{X}_0$. Then $(u, 1_X) \in A(\mathcal{X}_0 \times \mathcal{X}_0)$ and $(0_X, 1_X - v) \in A(\mathcal{X}_0 \times \mathcal{X}_0)$. Since $(1_X, v)$ is the complement of $(0_X, 1_X - v)$ and $(u, v) = (u, 1_X) \wedge (1_X, v)$, (u, v) belongs to each bold algebra containing $A(\mathcal{X}_0 \times \mathcal{X}_0)$ and the assertion follows.

Observe that Theorem 1 yields a one-to-one correspondence between objects of the form $A(\mathcal{X}_0 \times \mathcal{X}_0)$ and the bold algebras of the form $\mathcal{X}_0 \times \mathcal{X}_0$ and $\mathcal{M}(\mathbb{A}_{\mathcal{X}}) \times \mathcal{M}(\mathbb{A}_{\mathcal{X}})$, respectively.

PROBLEM. Consider a category $\mathcal{A}$, the objects of which are of the form $A(\mathcal{X}_0 \times \mathcal{X}_0)$, and the category category $\mathcal{B}$, the objects of which are of the form $\mathcal{X}_0 \times \mathcal{X}_0$. Having in mind relationships between the *IF* probability and the fuzzy probability, is it possible to define morphisms of $\mathcal{A}$ and $\mathcal{B}$ in a nontrivial way so that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic?

Let $\mathcal{X}_0 \subseteq I^X$, $\mathcal{Y}_0 \subseteq I^Y$ be bold algebras and let h be a sequentially continuous D-homomorphismm on $\mathcal{X}_0$ into $\mathcal{Y}_0$. For $(u,v) \in A(\mathcal{X}_0 \times \mathcal{X}_0)$ define $\overline{h}(u,v) = (h(u), h(v))$. It is easy to see that $\overline{h}(u,v) \in A(\mathcal{Y}_0 \times \mathcal{Y}_0)$. Denote $\overline{h}$ the resulting map on $A(\mathcal{X}_0 \times \mathcal{X}_0)$ into $A(\mathcal{Y}_0 \times \mathcal{Y}_0)$.

Solution

LEMMA 2. (i) $\overline{h}$ preserves the order and the Łukasiewicz operations: $\oplus, \odot$.
(ii) $\overline{h}$ is sequentially continuous.

Denote A the category having systems $A(\mathcal{X}_0 \times \mathcal{X}_0)$ as objects and maps $\overline{h}$ as morphisms. Further denote BID^2 the category having systems $\mathcal{X}_0 \times \mathcal{X}_0$ as objects and maps of the type $\overline{h}$ as morphisms.

THEOREM 3. The categories A and BID^2 are isomorphic.

Moreover, as stated on the previous talk, states on $A(\mathcal{M}(\mathbb{A}_{\mathcal{X}}) \times \mathcal{M}(\mathbb{A}_{\mathcal{X}}))$ and sequentially continuous D-homomorphisms on $\mathcal{M}(\mathbb{A}_{\mathcal{X}}) \times \mathcal{M}(\mathbb{A}_{\mathcal{X}})$ into I (states in the Fuzzy Probability Theory) are in one-to-one correspondence. Consequently, the IF-probability can be studied within FPT.

The category of products

In order to prove additional properties of powers of bold algebras, we embed BID^2 into the following category BID_2 of products:

(i) The objects are ordered pairs $(\mathcal{X}, \mathcal{Y})$, where $\mathcal{X} \subseteq [0,1]^X$ and $\mathcal{Y} \subseteq [0,1]^Y$ are bold algebras and the operations are defined coordinatewise;

(ii) The morphisms are ordered pairs (f, g) of sequentially continuous D-homomorphisms and the composition is parallel:

$$(f_2, g_2) \circ (f_1, g_1) = (f_2 \circ f_1, g_2 \circ g_1)$$

whenever the compositions $f_2 \circ f_1$ and $g_2 \circ g_1$ are defined.

THEOREM 4. Let $\mathcal{X}_0 \subseteq I^X$ and $\mathcal{Y}_0 \subseteq I^Y$ be bold algebras. Let $\mathcal{X} \subseteq I^X$, resp. $\mathcal{Y} \subseteq I^Y$, be the smallest sequentially closed bold algebra containing $\mathcal{X}_0$, resp. $\mathcal{Y}_0$. Then $\mathcal{M}(\mathbb{A}_{\mathcal{X}}) \times \mathcal{M}(\mathbb{A}_{\mathcal{Y}})$ is the epireflection of $\mathcal{X}_0 \times \mathcal{Y}_0$ into the subcategory *CGBID* of *BID*.

Hint of proof.

(i) Let h be a sequentially continuous D-homomorphisms on $\mathcal{X}_0 \times \mathcal{Y}_0$ into $\mathcal{M}(\mathbb{A})$. We have to prove that h can be uniquely extended to a sequentially continuous D-homomorphisms on $\mathcal{M}(\mathbb{A}_{\mathcal{X}}) \times \mathcal{M}(\mathbb{A}_{\mathcal{Y}})$ into $\mathcal{M}(\mathbb{A})$;

(ii) Let Z be the disjoint union of X and Y and let $\mathcal{Z}_0 \subseteq I^Z$ be the corresponding "disjoint union" of $\mathcal{X}_0$ and $\mathcal{Y}_0$. Then we can identify $\mathcal{M}(\mathbb{A}_{\mathcal{Z}})$ and $\mathcal{M}(\mathbb{A}_{\mathcal{X}}) \times \mathcal{M}(\mathbb{A}_{\mathcal{Y}})$.

(iii) Finally, (i) follows from the fact that $\mathcal{M}(\mathbb{A}_{\mathcal{Z}})$ is the epireflection of $\mathcal{X}_0 \times \mathcal{Y}_0$ into *CGBID*.

COROLLARY 5. The epireflector sending objects of *BID* into *CGBID* is productive.

COROLLARY 6. Let $\mathcal{X}_0 \subseteq I^X$ be a bold algebra and let h be a sequentially continuous D-homomorphismm on $\mathcal{X}_0$ into I. Then h can be uniquely extended to a sequentially continuous D-homomorphisms on $\mathcal{M}(\mathbb{A}_{\mathcal{X}}) \times \mathcal{M}(\mathbb{A}_{\mathcal{X}})$ into I.

This and the next theorem leads to two generalized probability theories which can be studied within FPT: one besed on BID^2 and the other based on BID_2.

Let $\mathcal{X}_0 \subseteq I^X$ and $\mathcal{Y}_0 \subseteq I^Y$ be bold algebras. Let g, h be sequentially continuous D-homomorphisms on $\mathcal{X}_0$, resp. on $\mathcal{Y}_0$, into $I = [0,1]$ and let $a \in [0,1]$. For $(u,v) \in \mathcal{X}_0 \times \mathcal{Y}_0$ put $(ag + (1-a)h)((u,v)) = ag(u) + (1-a)h(v)$ and denote $ag + (1-a)h$ the resulting map. It will be called a convex combination of g and h. The proof of the next lemma is straightforward and it is omitted.

LEMMA 7. $ag + (1-a)h$ is a sequentially continuous D-homomorphism.

THEOREM 8. Let $\mathcal{X}_0 \subseteq I^X$ and $\mathcal{Y}_0 \subseteq I^Y$ be bold algebras. Then the sequentially continuous D-homomorphisms on $\mathcal{X}_0 \times \mathcal{Y}_0$ into I are exactly the convex combinations of two sequentially continuous D-homomorphisms on $\mathcal{X}_0$, resp. on $\mathcal{Y}_0$, into I.