Quasicontinuous functions and points of uniform convergence

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Definition of $LU(f_n)$

X - topological space, (Y, d) - metric space, $f_n : X \to Y$

Definition (Drahovský, Š., Šalát, T., Toma, V.: *Points of uniform convergence and oscillation of sequences of functions*, Real Anal. Exchange **20** (1994/95), 753–767.)

We say that the sequence $(f_n)_n$ locally uniformly converges at a point $x \in X$ if there is a neighbourhood U of x such that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(f_n(y), f_m(y)) < \varepsilon$ for each $y \in U$ and each $m, n \geq n_0$.

$LU(f_n)$ - points of locally uniform convergence

The set $LU(f_n)$

If $f_n \to f$ then $LU(f_n) = \{x \in X : \exists \; open \; U \ni x; f_n \upharpoonright U \rightrightarrows f \upharpoonright U\}$

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If X is compact, $(f_n)_n$ converges to f and $LU(f_n) = X$ then $(f_n)_n$ uniformly converges to f.

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$f_n \to f$ and f_n continuous at $x \in LU(f_n)$ then f continuous at x

Characterization of $LU(f_n)$

Theorem (Drahovský, Š., Šalát, T., Toma, V.: *Points of uniform convergence and oscillation of sequences of functions*, Real Anal. Exchange **20** (1994/95), 753–767.)

$LU(f_n)$ is an open set.

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Theorem

Let X be a topological space and let G be an open set. There there are functions $f_n : X \to \mathbb{R}$ such that $G = LU(f_n)$.

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Theorem

Let X be a separable metric space and let G be an open set containing all isolated points. Then there is a convergent sequence $(f_n)_n$ of continuous functions $f_n : X \to \mathbb{R}$ such that $G = LU(f_n)$.

X - topological space, (Y, d) - metric space, $f_n : X \to Y$

Definition (Goffman, C.: *Reelle Funktionen*, Bibliographishes Institut, Manheim-Wien,Zürich, 1976.)

We say that the sequence $(f_n)_n$ uniformly converges at a point $x \in X$ if for every $\varepsilon > 0$ there is a neighbourhood U of x and $n_0 \in \mathbb{N}$ such that $d(f_n(y), f_m(y)) < \varepsilon$ for each $y \in U$ and each $m, n \geq n_0$.

$U(f_n)$ - points of uniform convergence

The set $U(f_n)$

If $f_n \to f$ then $U(f_n) = \{x \in X : \forall \epsilon > 0 \ \exists \ open \ U \ni x \ \exists n_0 \in \mathbb{N} \ \forall n \geq n_0 \ \forall y \in U : d(f_n(y), f(y)) < \varepsilon\}$

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$f_n \to f$ and f_n continuous at $x \in U(f_n)$ then f continuous at x

[DST]: $LU(f_n) \subset U(f_n)$

Sets $LU(f_n)$ and $U(f_n)$

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[DST]: there is a convergent sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$: $LU(f_n) = \emptyset$ and $U(f_n)$ is residual

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If $U(f_n) = \mathbb{R}$ then $LU(f_n) = \mathbb{R}$

Characterization of $U(f_n)$

Theorem (Drahovský, Š., Šalát, T., Toma, V.: *Points of uniform convergence and oscillation of sequences of functions*, Real Anal. Exchange **20** (1994/95), 753–767.)

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Theorem

Let X be a metric space and $G \subset X$. Then $G = U(f_n)$ for some sequence of functions $f_n : X \to \mathbb{R}$ if and only if G is a G_δ-set.

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$f : X \to Y$ is quasicontinuous at $x \in X$ if for each neighbourhood V of $f(x)$ and each neighbourhood U of x there is an open nonempty set $G \subset U$ such that $f(G) \subset V$.

$Q(f)$ - points of quasicontinuity of f

Characterization of $U(f_n)$

Theorem

Let X be a metric space. Let G be a G_δ-set containing all isolated points. There there are functions $f_n : X \to \mathbb{R}$ such that $(f_n)_n$ converges to some f and $G = U(f_n)$.

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Let X be a metric space. Let G be a G_δ-set. Then there are quasicontinuous functions $f_n : X \to \mathbb{R}$ such that $G = U(f_n)$.

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Theorem

Let X be a metric space. Let G be a G_δ-set. Then there are quasicontinuous functions $f_n : X \to \mathbb{R}$ such that $G = U(f_n)$.

However, we cannot require simultaneously that f_n are quasicontinuous and $(f_n)_n$ converges.

Theorem

Let X be a Baire space. Let $f_n : X \to \mathbb{R}$ be quasicontinuous functions converging to some function $f : X \to \mathbb{R}$. Then the set $U(f_n)$ is dense.

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Theorem

Let X be a metric space. Then the following are equivalent:
(a) X is Baire;
(b) If $f_n : X \to \mathbb{R}$ are quasicontinuous functions and $f_n \to f$ then the set $U(f_n)$ is dense.

Theorem

Let X be a Baire space and let (Y, d) be a metric space. Let $f_n : X \to Y$ be quasicontinuous functions and let $f : X \to Y$ be a pointwise discontinuous function such that $Gr(f) \subset \mathrm{Li}\, Gr(f_n)$. Then the set $U(f_n)$ is dense in X.

Definition of $E(f_n)$

Definition (Holá, Ľ., Holý, D.: *Pointwise convergence of quasicontinuous functions and Baire spaces*, Rocky Mountain J. Math. **41** (2011), 1883–1894.)

We say that the sequence $(f_n)_n$ is equi-quasicontinuous at a point $x \in X$ if for every $\varepsilon > 0$ and for every neighbourhood U of x there is $n_0 \in \mathbb{N}$ and a nonempty open set $G \subset U$ such that $d(f_n(y), f_n(x)) < \varepsilon$ for each $y \in G$ and each $n \geq n_0$.

$E(f_n)$ - points of equi-quasicontinuity

Characterization of $E(f_n)$

Theorem

The set $E(f_n)$ is the countable intersection of decreasing sequence of semi-open sets.

A is semi-open if $A \subset \operatorname{Cl}\operatorname{Int} A$

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The set $E(f_n)$ is the countable intersection of decreasing sequence of semi-open sets.

A is semi-open if $A \subset \operatorname{Cl}\operatorname{Int} A$
X is resolvable if X is the union of two disjoint dense sets

Theorem

Let X be a resolvable space. Let E be a subset of X. Then $E = E(f_n)$ for some $f_n : X \to \mathbb{R}$ if and only if $E = \bigcap_{n\in\mathbb{N}} E_n$, where E_n are semi-open and $E_{n+1} \subset E_n$.

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If f_n are quasicontinuous and $f_n \to f$ then $U(f_n) \subset Q(f)$.

Theorem

Let X be a Baire space. If $E(f_n)$ is dense then $U(f_n)$ is dense.

X Baire and sets $U(f_n)$ and $E(f_n)$

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F1. f_n, f quasicontinuous, $f_n \to f$: $E(f_n) = X$, $U(f_n)$ is dense (it can be $\operatorname{Int} U(f_n) = \emptyset$).

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F2. f_n quasicontinuous, $f_n \to f$: $E(f_n)$ and $U(f_n)$ are dense, $U(f_n) \subset E(f_n)$ (it can be $\operatorname{Int} E(f_n) = \operatorname{Int} U(f_n) = \emptyset$).

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F3. f_n quasicontinuous: $U(f_n) \subset E(f_n)$ (it can be $E(f_n) = U(f_n) = \emptyset$).

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F3. f_n quasicontinuous: $U(f_n) \subset E(f_n)$ (it can be $E(f_n) = U(f_n) = \emptyset$).

F4. $f_n \to f$: it may be $E(f_n) = U(f_n) = \emptyset$ or $U(f_n) \not\subset E(f_n)$.