Symmetrization and convexity II

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Introduction

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The Hermite–Hadamard inequality

$f : [a, b] \to \mathbb{R}$ – convex

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)}{2}$$

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Wright-convexity

$D \subset \mathbb{R}^n$ – a convex set

$f : D \to \mathbb{R}$ is called *Wright-convex* (W-convex), if

$$f(tx + (1-t)y) + f((1-t)x + ty) \leqslant f(x) + f(y)$$

for any $x, y \in D$ and $t \in [0, 1]$.

- convexity $\Rightarrow$ W-convexity
- W-convexity $\not\Rightarrow$ convexity

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Ng's representation for W-convex functions

$D \subset \mathbb{R}^n$ – open and convex, $f : D \to \mathbb{R}$ is W-convex iff

$$f(x) = g(x) + a(x), \quad x \in D,$$

where $g : D \to \mathbb{R}$ is convex, $a : \mathbb{R}^n \to \mathbb{R}$ is additive.

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Inequality of Hermite–Hadamard type (Olbryś 2013)

$\mathcal{I} \subset \mathbb{R}$ – open interval

if $f : \mathcal{I} \to \mathbb{R}$ is W-convex, then

$$2f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b \big(f(x) + f(a+b-x)\big)\,dx \leqslant f(a) + f(b)$$

for any $a, b \in \mathcal{I}$.

$v_0, \ldots, v_n \in \mathbb{R}^n$ – affinely independent

$S = \mathrm{conv}\{v_0, \ldots, v_n\}$ – simplex with vertices $v_0, \ldots, v_n$

- $|S| = vol(S)$

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$$\mathbf{b} = \frac{1}{n+1} \sum_{i=0}^{n} v_i$$

- any element $x \in S$ is uniquely represented by a convex combination of the vertices:

$$x = \sum_{i=0}^{n} t_i v_i$$

where $t_i \geqslant 0$, $i = 0, \ldots, n$, with $t_0 + \cdots + t_n = 1$

C – set of all cyclic permutations of $\{0, \ldots, n\}$.
$\sigma \in C$, $f : S \to \mathbb{R}$

$$f_\sigma(x) = f\big(\sigma(x)\big)$$

Symmetrization

$$F(x) = \sum_{\sigma \in C} f_\sigma(x)\,, \quad x \in S$$

F is symmetric with respect to the barycenter

$$F\big(\sigma(x)\big) = F(x), \qquad \sigma \in C$$

Theorem 1

If $f : S \to \mathbb{R}$ is convex then

$$f(\mathbf{b}) \leqslant \frac{1}{|S|} \int_S f(x)\, dx \leqslant \frac{1}{n+1} \sum_{i=0}^{n} f(v_i).$$

- E. Neuman, *Inequalities involving multivariate convex functions II*, Proc. Amer. Math. Soc. **109** (1990), 965–974.
- M. Bessenyei, *The Hermite–Hadamard inequality on simplices*, Amer. Math. Monthly **115** (2008), 339–345.
- Sz. Wąsowicz, *Hermite–Hadamard-type inequalities in the approximate integration*, Math. Inequal. Appl. **11** (2008), 693–700.

Hermite–Hadamard type inequality for W-convex functions

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Theorem 2

$D \subset \mathbb{R}^n$ open and convex, $S \subset D$ – a simplex.
If $f : D \to \mathbb{R}$ is W-convex, then its symmetrization F is convex on S.

Lemma 1 (Wąsowicz, Witkowski)

If $g : S \to R$ is convex then so is g_σ.

Lemma 2

If $a : \mathbb{R}^n \to \mathbb{R}$ is additive, then its symmetrization $A = \sum_{\sigma \in C} a_\sigma$ is constant.

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$$F(x) = \sum_{\sigma \in C} f_\sigma(x)\,, \quad x \in S.$$

$$F(\mathbf{b}) \leqslant \frac{1}{|S|} \int_S F(x)\,dx \leqslant \frac{1}{n+1} \sum_{i=0}^{n} F(\nu_i).$$

Theorem 3

If $f : D \to \mathbb{R}$ is W-convex, then

$$(n+1) f(\mathbf{b}) \leqslant \frac{1}{|S|} \int_S \Big(\sum_{\sigma \in C} f_\sigma(x) \Big)\,dx \leqslant \sum_{i=0}^{n} f(\nu_i).$$

Remark. For $n = 1$ and $S = [a, b]$ we obtain as a consequence inequality due to Olbryś.

Hermite–Hadamard type inequality for strongly convex functions

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Strong convexity with modulus c

$D \subset \mathbb{R}^n$ – a convex subset of an inner product space, $c > 0$.

$f : D \to \mathbb{R}$ is called *strongly convex with modulus* c, if

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y) - ct(1-t)\|x-y\|^2$$

for all $x, y \in D$ and $t \in [0, 1]$.

- strong convexity $\Rightarrow$ convexity

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- convexity $\not\Rightarrow$ strong convexity

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- strong convexity $\Rightarrow$ convexity
- convexity $\not\Rightarrow$ strong convexity
- f – strongly convex $\iff$ $f - c\|\cdot\|^2$ – convex

Hermite–Hadamard type inequality for strongly convex functions

By the Hermite–Hadamard inequality (cf. Theorem 1 with convex function $g = f - c\|\cdot\|^2$) we get

Theorem 4

If $f : S \to \mathbb{R}$ is strongly convex with modulus c, then

$$f(\mathbf{b}) + c\left(\frac{1}{|S|}\int_S \|x\|^2\,dx - \|\mathbf{b}\|^2\right) \leqslant \frac{1}{|S|}\int_S f(x)\,dx$$
$$\leqslant \frac{1}{n+1}\sum_{i=0}^{n} f(v_i) + c\left(\frac{1}{|S|}\int_S \|x\|^2\,dx - \frac{1}{n+1}\sum_{i=0}^{n}\|v_i\|^2\right).$$

It is the multivariate counterpart of a result due to Merentes and Nikodem (*Remarks on strongly convex functions*, Aequationes Math. **80** (2010), 193–199).

S_1 – the unit simplex in $\mathbb{R}^n$, i.e. the simplex with vertices $e_0 = (0, 0, \dots, 0)$, $e_1 = (1, 0, \dots, 0)$, …, $e_n = (0, \dots, 0, 1)$.

Corollary 1

If $f : S_1 \to \mathbb{R}$ is strongly convex with modulus c, then

$$f\left(\frac{1}{n+1}\sum_{i=0}^{n} e_i\right) + \frac{cn^2}{(n+1)^2(n+2)} \leqslant n! \int_{S_1} f(x)\,dx \leqslant \frac{1}{n+1}\sum_{i=0}^{n} f(e_i) - \frac{cn^2}{(n+1)(n+2)}.$$

Remark. For $n = 1$ we get $f\left(\frac{1}{2}\right) + \frac{c}{12} \leqslant \int_0^1 f(x)\,dx \leqslant \frac{f(0)+f(1)}{2} - \frac{c}{6}$, which corresponds to the result by Merentes and Nikodem

Hermite–Hadamard type inequality for strongly W-convex functions

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Strong W-convexity

$f : D \to \mathbb{R}$ is called *strongly W-convex with modulus* c, if

$$f(tx+(1-t)y)+f((1-t)x+ty) \leqslant f(x)+f(y)-2ct(1-t)\|x-y\|^2$$

for all $x, y \in D$ and $t \in [0,1]$.

Theorem 5

If $f : D \to \mathbb{R}$ is strongly W-convex with modulus c, then its symmetrization F is strongly convex on S with modulus $(n+1)c$. In particular, F is integrable on S.

F – strongly convex on S with modulus $(n+1)c$. By virtue of Theorem 4

Corollary 2

If $f : S \to \mathbb{R}$ is strongly W-convex with modulus c, then

$$f(\mathbf{b}) + c\left(\frac{1}{|S|}\int_S \|x\|^2\,dx - \|\mathbf{b}\|^2\right) \leqslant \frac{1}{(n+1)|S|}\int_S \left(\sum_{\sigma\in C} f_\sigma(x)\,dx\right)$$

$$\leqslant \frac{1}{n+1}\sum_{i=0}^{n} f(v_i) + c\left(\frac{1}{|S|}\int_S \|x\|^2\,dx - \frac{1}{n+1}\sum_{i=0}^{n}\|v_i\|^2\right)$$

Corollary 3

If $f : S_1 \to \mathbb{R}$ is strongly W-convex with modulus c, then

$$f(\mathbf{b}) + \frac{cn^2}{(n+1)^2(n+2)} \leqslant \frac{n!}{(n+1)} \int_{S_1} \left(\sum_{\sigma \in C} f_\sigma(x)\, dx \right) \leqslant \frac{1}{n+1} \sum_{i=0}^{n} f(v_i) - \frac{cn^2}{(n+1)(n+2)}.$$

Remark. For $n = 1$ we get

$$f\left(\frac{1}{2}\right) + \frac{c}{12} \leqslant \frac{1}{2} \int_0^1 (f(x) + f(1-x))\, dx \leqslant \frac{f(0)+f(1)}{2} - \frac{c}{6}$$

Thank you very much for your kind attention