Ideal quasi-normal convergence

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All functions are assumed to be real-valued.

All topological spaces are assumed to be Hausdorff.

Quasi-normal (equal) convergence

Convergence of $\langle f_n;\ n \in \omega \rangle$, $\quad f_n, f : X \to \mathbb{R}$

Pointwise convergence $\qquad f_n \to f$

$$(\forall x \in X)(\forall \varepsilon > 0)(\exists n_0)(\forall n \in \omega)(n \geq n_0 \to |f_n(x) - f(x)| < \varepsilon)$$

Quasi-normal convergence $\qquad f_n \xrightarrow{\text{QN}} f$

there exists $\{\varepsilon_n\}_{n=0}^{\infty}$ converging to 0 such that

$$(\forall x \in X)(\exists n_0)(\forall n \in \omega)(n \geq n_0 \to |f_n(x) - f(x)| < \varepsilon_n)$$

Theorem

If X is a perfect separable metric space then there is a sequence $\langle f_n;\ n \in \omega \rangle$ of continuous functions on X such that $f_n \to 0$ but no subsequence of $\langle f_n;\ n \in \omega \rangle$ does converge quasi-normally.

see Bukovský L., The Structure of the Real Line, Monogr. Mat., Springer-Birkhauser, Basel, 2011.

Convergence of $\langle f_n;\ n \in \omega \rangle$, $\quad f_n, f : X \to \mathbb{R}$

Bukovská Z., *Thin sets in trigonometrical series and quasinormal convergence,* Math. Slovaca **40** (1990), 53–62.

Theorem

For any set X with $|X| < \mathfrak{b}$ and for any $f_n, f : X \to \mathbb{R}$, if $f_n \to f$ on X then $f_n \xrightarrow{\text{QN}} f$.

Bukovský L., Recław I. and Repický M., *Spaces not distinguishing pointwise and quasinormal convergence of real functions,* Topology Appl. **41** (1991), 25–40.

Theorem

There is a subset of reals X of cardinality $\mathfrak{b}$ and a sequence of continuous functions $\langle f_n;\ n \in \omega \rangle$ on X such that $f_n \to 0$ but $\langle f_n;\ n \in \omega \rangle$ does not converge quasi-normally.

Bukovský L., Recław I. and Repický M., *Spaces not distinguishing pointwise and quasinormal convergence of real functions,* Topology Appl. **41** (1991), 25–40.

A topological space X is a QN-space if each sequence of continuous real-valued functions converging to zero on X is converging quasi-normally.

- Tychonoff QN-space is zero-dimensional
- any QN-subset of a metric separable space is perfectly meager
- perfectly normal QN-space has Hurewicz property
- $\mathrm{non}(\text{QN-space}) = \mathfrak{b}$
- $\mathfrak{b}$-Sierpiński set is a QN-space (exists under $\mathfrak{b} = \mathrm{cov}(\mathcal{N}) = \mathrm{cof}(\mathcal{N})$)

Recław I., *Metric spaces not distinguishing pointwise and quasinormal convergence of real functions,* Bull. Acad. Polon. Sci. **45** (1997), 287–289.

Miller A.W., *On the length of Borel hierarchies,* Ann. Math. Logic **16** (1979), 233–267.

- perfectly normal QN-space is a σ-set
- the theory **ZFC** + "any QN-space is countable" is consistent

Ideal versions

A family $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is called an ideal if

a) $B \in \mathcal{I}$ for any $B \subseteq A \in \mathcal{I}$,
b) $A \cup B \in \mathcal{I}$ for any $A, B \in \mathcal{I}$,
c) $\mathrm{Fin} = [\omega]^{<\omega} \subseteq \mathcal{I}$,
d) $\omega \notin \mathcal{I}$.

$\mathcal{I} \subseteq \mathcal{P}(Z)$

$[Z]^{<\omega} \subseteq \mathcal{I}$

$Z \notin \mathcal{I}$

$\mathcal{I}, \mathcal{J}$ are ideals in the following.

$$\mathcal{A} \subseteq \mathcal{P}(\omega) \qquad \mathcal{A}^d = \{A \subseteq \omega;\ \omega \setminus A \in \mathcal{A}\}$$

A family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is called a filter if $\mathcal{F}^d$ is ideal.

A maximal filter is called ultrafilter.

Cartan H., *Filtres et ultrafiltres,* C. R. Acad. Sci. Paris **205** (1937), 777–779.

Kostyrko P., Šalát T. and Wilczyński W., *$\mathcal{I}$-convergence,* Real Anal. Exchange **26** (2000/2001), 669–685.

$\mathcal{I}$-convergence of reals $x_n \xrightarrow{\mathcal{I}} x$

$$(\forall \varepsilon > 0)(\exists A \in \mathcal{I})(\forall n \in \omega)(n \notin A \to |x_n - x| < \varepsilon)$$

$\mathcal{I}$-pointwise convergence $f_n \xrightarrow{\mathcal{I}} f$

$$(\forall x \in X)(\forall \varepsilon > 0)(\exists A \in \mathcal{I})(\forall n \in \omega)(n \notin A \to |f_n(x) - f(x)| < \varepsilon)$$

Das P. and Chandra D., *Spaces not distinguishing pointwise and $\mathcal{I}$-quasinormal convergence of real functions,* Comment. Math. Univ. Carolin. **54** (2013), 83–96.

$\mathcal{I}$-quasi-normal convergence $f_n \xrightarrow{\mathcal{I}\text{QN}} f$

there exists $\{\varepsilon_n\}_{n=0}^{\infty}$ $\mathcal{I}$-converging to 0 such that

$$(\forall x \in X)(\exists A \in \mathcal{I})(\forall n \in \omega)(n \notin A \to |f_n(x) - f(x)| < \varepsilon_n)$$

Das P. and Chandra D., *Spaces not distinguishing pointwise and $\mathcal{I}$-quasinormal convergence of real functions,* Comment. Math. Univ. Carolin. **54** (2013), 83–96.

A topological space X is a $\mathcal{J}$QN-space if each sequence of continuous functions converging to zero on X is converging $\mathcal{J}$-quasi-normally.

$$\boxed{\text{QN} \to \mathcal{J}\text{QN}}$$

Das P. and Chandra D., *$(\mathcal{I},\mathcal{J})$-quasinormal spaces,* manuscript.

A topological space X is an $(\mathcal{I},\mathcal{J})$QN-space if each sequence of continuous functions $\mathcal{I}$-converging to zero on X is converging $\mathcal{J}$-quasinormally.

Nonweak P-ideal

$A, B \subseteq \omega$ $\qquad$ $A \subseteq^* B$ if $A \setminus B \in \mathrm{Fin}$

Ideal $\mathcal{J}$ is a P-ideal if

- for any $\{A_n;\ n \in \omega\} \subseteq \mathcal{J}$ there is $A \in \mathcal{J}$ such that $A_n \subseteq^* A$ for each n.
- for any $\{A_n;\ n \in \omega\} \subseteq \mathcal{J}^d$ there is $A \in \mathcal{J}^d$ such that $A \subseteq^* A_n$ for each n.

$\mathcal{J}$ a P-ideal - $\mathcal{J}^d$ a P-filter

ultrafilter $\mathcal{U}$ a P-filter - a P-point

Filipów R. and Staniszewski M., *On ideal equal convergence*, Cent. Eur. J. Math **12** (2014), 896–910.

Mačaj M. and Sleziak M., *$\mathcal{I}^{\mathcal{K}}$-convergence*, Real Anal. Exchange **36** (2010/11), 177–194.

$A, B \subseteq \omega$ $\qquad$ $A \subseteq^{\mathcal{I}} B$ if $A \setminus B \in \mathcal{I}$

Ideal $\mathcal{J}$ is a P($\mathcal{I}$)-ideal if

- for any $\{A_n;\ n \in \omega\} \subseteq \mathcal{J}$ there is $A \in \mathcal{J}$ such that $A_n \subseteq^{\mathcal{I}} A$ for each n.
- for any $\{A_n;\ n \in \omega\} \subseteq \mathcal{J}^d$ there is $A \in \mathcal{J}^d$ such that $A \subseteq^{\mathcal{I}} A_n$ for each n.

$$\boxed{\mathcal{I}^+ = \mathcal{P}(\omega) \setminus \mathcal{I}}$$

Ideal $\mathcal{J}$ is a weak P-ideal if for any $\{A_n;\ n \in \omega\} \subseteq \mathcal{J}^d$ there is $A \in \mathcal{J}^+$ such that $A \subseteq^* A_n$ for each n.

$\boxed{\mathcal{J} \text{ a weak P-ideal} \quad - \quad \mathcal{J}^d \text{ a weak P-filter}}$

$\boxed{\text{ultrafilter } \mathcal{U} \text{ a weak P-filter} \quad - \quad \text{a P-point}}$

Ideal $\mathcal{J}$ is a weak P($\mathcal{I}$)-ideal if for any $\{A_n;\ n \in \omega\} \subseteq \mathcal{J}^d$ there is $A \in \mathcal{J}^+$ such that $A \subseteq^{\mathcal{I}} A_n$ for each n.

Proposition

Let $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(\omega)$ be ideals. The following statements are equivalent.

(1) *$\mathcal{J}$ is a weak* P($\mathcal{I}$)*-ideal.*

(2) *For any family $\{A_n;\ n \in \omega\} \subseteq \mathcal{J}$ there is $A \in \mathcal{J}^+$ such that $A \cap A_n \in \mathcal{I}$ for any $n \in \omega$.*

(3) *For any partition $\{I_n;\ n \in \omega\} \subseteq \mathcal{J}$ of ω there is $I \in \mathcal{J}^+$ such that $I \cap I_n \in \mathcal{I}$ for any $n \in \omega$.*

Proposition

Let $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(\omega)$ be ideals. The following statements are equivalent.

(1) *$\mathcal{J}$ is not a weak* P($\mathcal{I}$)*-ideal.*

(2) *There is a partition $\{I_n;\ n \in \omega\} \subseteq \mathcal{J}$ of ω such that for any $\{B_n;\ n \in \omega\} \subseteq \mathcal{I}$ we have $\bigcup_{n\in\omega} (B_n \cap I_n) \in \mathcal{J}$.*

(3) *For any cardinal κ and for any $\{\langle E_n^\alpha;\ n \in \omega\rangle;\ \alpha < \kappa\} \subseteq {}^\omega\mathcal{I}$ with $E_k^\alpha \cap E_m^\alpha = \emptyset, n \neq m, \alpha < \kappa$ there is a partition $\{I_n;\ n \in \omega\} \subseteq \mathcal{J}$ of ω such that for any $\alpha < \kappa$ we have*

$$\bigcup_{n\in\omega} \left(\bigcup_{i<n} E_i^\alpha \cap I_n \right) \in \mathcal{J}.$$

$$\boxed{\mathcal{I} \not\subset \mathcal{J} \rightarrow \mathcal{J} \text{ is a weak P}(\mathcal{I})\text{-ideal}}$$

$$\boxed{\mathcal{J} \text{ is not a weak P}(\mathcal{I})\text{-ideal} \rightarrow \mathcal{I} \subset \mathcal{J}}$$

Filipów R. and Szuca P., *Three kinds of convergence and the associated $\mathcal{I}$-Baire classes*, J. Math. Anal. Appl. **391** (2012), 1–9.

Laczkovich M. and Recław I., *Ideal limits of sequences of continuous functions*, Fund. Math. **203** (2009), 39–46.

Laflamme C., *Filter games and combinatorial properties of winning strategies*, Contemp. Math. **192** (1996), 51–67.

The following statements are equivalent.

(1) $\mathcal{J}$ is not a weak P-ideal.

(2) $\mathrm{Fin}\times\mathrm{Fin} \leq_{\mathrm{K}} \mathcal{J}$.

(3) $\mathrm{Fin}\times\mathrm{Fin} \leq_{\mathrm{KB}} \mathcal{J}$.

(4) $\mathrm{Fin}\times\mathrm{Fin} \sqsubseteq \mathcal{J}$.

(5) Player I has a winning strategy in a game $G(\mathcal{J})$.

$$\boxed{\mathrm{Fin}\times\mathrm{Fin} = \{A\subseteq\omega\times\omega;\ \{n;\ \{m;\ (n,m)\in A\}\notin\mathrm{Fin}\}\in\mathrm{Fin}\}}$$

Player I in the nth move plays an element $C_n\in\mathcal{J}$, and then player II plays $F_n\in[\omega\setminus C_n]^{<\omega}$. Player I wins when $\bigcup_{n\in\omega}F_n\in\mathcal{J}$.

$\mathcal{I}\leq_{\mathrm{K}}\mathcal{J}$ if there is a function $\varphi:\omega\to\omega\times\omega$ such that $\varphi^{-1}(I)\in\mathcal{J}$ for any $I\in\mathcal{I}$.

$\mathcal{I}\leq_{\mathrm{KB}}\mathcal{J}$ if … a finite-to-one function … .

$\mathcal{J}$ contains an isomorphic copy of ideal $\mathcal{I}$ (shortly $\mathcal{I}\sqsubseteq\mathcal{J}$) if there is a bijection $\varphi:\omega\to\omega\times\omega$ such that $\varphi^{-1}(A)\in\mathcal{J}$ for any $A\in\mathcal{J}$.

Theorem

Let $\mathcal{I},\mathcal{J}\subseteq\mathcal{P}(\omega)$ be ideals such that $\mathcal{J}$ is not a weak P($\mathcal{I}$)*-ideal. There is a sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ $\mathcal{J}$-converging to zero such that if $\langle f_n;\ n\in\omega\rangle$ is a sequence of functions on some set X and $f_n\xrightarrow{\mathcal{I}} f$ then $f_n \xrightarrow{\mathcal{J}\mathrm{QN}} f$ with the control $\{\varepsilon_n\}_{n=0}^{\infty}$.*

Corollary

Let $\mathcal{J}\subseteq\mathcal{P}(\omega)$ be an ideal which is not a weak P*-ideal. There is a sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ $\mathcal{J}$-converging to zero such that if $\langle f_n;\ n\in\omega\rangle$ is a sequence of functions on some set X and $f_n\to f$ then $f_n\xrightarrow{\mathcal{J}\mathrm{QN}} f$.*

Corollary

Let $\mathcal{I},\mathcal{J}\subseteq\mathcal{P}(\omega)$ be ideals.

(i) *If $\mathcal{J}$ is not a weak* P($\mathcal{I}$)*-ideal then any topological space is an* ($\mathcal{I}$,$\mathcal{J}$)QN*-space.*

(ii) *If $\mathcal{J}$ is not a weak* P*-ideal then any topological space is a $\mathcal{J}$*QN*-space.*

Shelah S., Proper forcing, Springer-Verlag, 1982.

Corollary

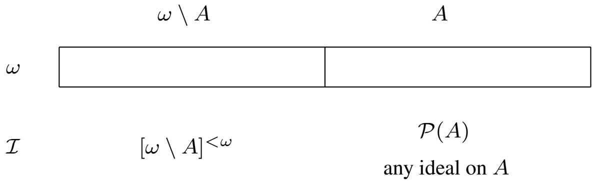
The theory **ZFC** + *"any topological space is a $\mathcal{U}^d$*QN*-space for any ultrafilter $\mathcal{U}\subseteq\mathcal{P}(\omega)$" is consistent (relatively to* **ZFC***).*

Ideals with a pseudounion

A set $A \subseteq \omega$ is called a pseudounion of the family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ if $\omega \setminus A$ is infinite and $B \subseteq^* A$ for any $B \in \mathcal{A}$.

Thus an ideal $\mathcal{I}$ is a P-ideal if and only if every countable subfamily of $\mathcal{I}$ has a pseudounion belonging to $\mathcal{I}$.

If a pseudounion A of $\mathcal{I}$ belongs to $\mathcal{I}$ then $\mathcal{I} = \{B \subseteq \omega;\ B \subseteq^* A\}$.



Let $\mathcal{I} \subseteq \mathcal{P}(\omega)$ be an ideal. The following statements are equivalent.

(1) $\mathcal{I}$ has a pseudounion.

(2) $\mathcal{I}$ is not tall.

(3) $\mathcal{I} \leq_{\mathrm{K}}$ Fin.

(4) $\mathcal{I} \leq_{\mathrm{KB}}$ Fin.

(5) $\mathcal{I} \leq_{\mathrm{RB}}$ Fin.

If $\mathrm{cof}(\mathcal{I}) < \mathfrak{p}$ then $\mathcal{I}$ has a pseudounion.

$\emptyset \times \mathrm{Fin}$ has a pseudounion and $\mathrm{cof}(\emptyset \times \mathrm{Fin}) = \mathfrak{d}$.

$$\emptyset \times \mathrm{Fin} = \{A \subseteq \omega \times \omega;\ (\forall n \in \omega)\ \{m;\ (n, m) \in A\} \in \mathrm{Fin}\}$$

The n-th element of $A \subseteq \omega$ is denoted $\mathrm{e}_A(n)$.

Proposition (L. Bukovský – P. Das – J. Š.)

Let C be a pseudounion of an ideal $\mathcal{I}$, $A = \omega \setminus C$. Then

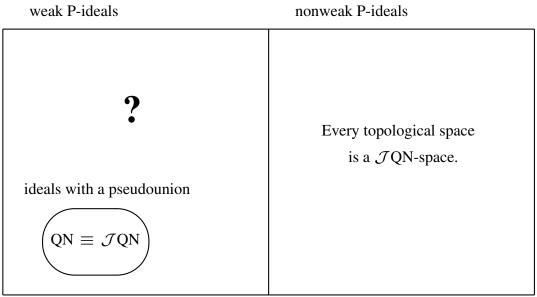
a) *For any sequence $\langle f_n;\ n \in \omega \rangle$ of real-valued functions on X, if $f_n \xrightarrow{\mathcal{I}} f$ then $f_{e_A(n)} \to f$.*

b) *For any sequence $\langle f_n;\ n \in \omega \rangle$ of real-valued functions on X, if $f_n \xrightarrow{\mathcal{I}\mathrm{QN}} f$ then $f_{e_A(n)} \xrightarrow{\mathrm{QN}} f$.*

Corollary (L. Bukovský – P. Das – J. Š.)

Let $\mathcal{I} \subseteq \mathcal{P}(\omega)$ be an ideal with a pseudounion. Then any topological space X is an $\mathcal{I}$QN-space if and only if X is a QN-space.

Other ideals


weak P-ideals
nonweak P-ideals
?
ideals with a pseudounion
$\mathrm{QN} \equiv \mathcal{J}\mathrm{QN}$
Every topological space
is a $\mathcal{J}$QN-space.

$\boxed{\mathcal{J} \text{ a weak P}(\mathcal{I})\text{-ideal}}$

$$\kappa(\mathcal{I},\mathcal{J}) = \min\{|\mathcal{A}|;\ \mathcal{A} \subseteq {}^{\omega}\mathcal{I} \wedge$$
$$(\forall\{I_n;\ n\in\omega\} \subseteq \mathcal{J}, \text{part. of } \omega)(\exists\langle B_n;\ n\in\omega\rangle \in \mathcal{A}) \bigcup_{n\in\omega}(B_n \cap I_n) \in \mathcal{J}^+\}$$

Filipów R. and Staniszewski M., *Pointwise versus equal (quasi-normal) convergence via ideals*, manuscript.

$$\boxed{\mathfrak{b}(\mathcal{I},\mathcal{I}) = \kappa(\mathcal{I},\mathcal{I})}$$

$\boxed{\mathcal{I} \subseteq \mathcal{J} \to \aleph_1 \leq \kappa(\mathcal{I},\mathcal{J}) \leq \mathfrak{c}}$ $\boxed{\mathcal{I} \not\subseteq \mathcal{J} \to \kappa(\mathcal{I},\mathcal{J}) = 1}$

Theorem

Let $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(\omega)$ *be ideals such that* $\mathcal{J}$ *is a weak* $\mathrm{P}(\mathcal{I})$*-ideal,* X *being a set. The following statements are equivalent.*

(1) $|X| < \kappa(\mathcal{I}, \mathcal{J})$.

(2) *For any sequence* $\langle f_n;\ n \in \omega\rangle$ *of functions on* X*, if* $f_n \xrightarrow{\mathcal{I}} f$ *then* $f_n \xrightarrow{\mathcal{J}\mathrm{QN}} f$.

Corollary

Let $\mathcal{J} \subseteq \mathcal{P}(\omega)$ *be a weak* P*-ideal,* X *being a set. The following statements are equivalent.*

(1) $|X| < \kappa(\mathrm{Fin}, \mathcal{J})$.

(2) *For any sequence* $\langle f_n;\ n \in \omega\rangle$ *of functions on* X*, if* $f_n \to f$ *then* $f_n \xrightarrow{\mathcal{J}\mathrm{QN}} f$.

non($\mathcal{J}$QN-space)/non(($\mathcal{I}$,$\mathcal{J}$)QN-space) denotes the minimal cardinality of a perfectly normal space which is not an $\mathcal{J}$QN-space/an ($\mathcal{I}$,$\mathcal{J}$)QN-space.

Corollary

Let $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(\omega)$ be ideals.

(i) *If $\mathcal{J}$ is a weak* P*-ideal then* non($\mathcal{J}$QN-*space*) $= \kappa(\mathrm{Fin}, \mathcal{J}) \geq \mathfrak{b}$.

(ii) *If $\mathcal{J}$ is a weak* P($\mathcal{I}$)*-ideal then non*(($\mathcal{I}$,$\mathcal{J}$)QN-space) $= \kappa(\mathcal{I}, \mathcal{J})$.

Strong $\mathcal{J}$QN-space

$\mathrm{e}_A(n)$ denotes the n-th element of $A \subseteq \omega$.

A topological space X is **a strong $\mathcal{J}$QN-space**, shortly s$\mathcal{J}$QN-space, if for any sequence $\langle f_n;\ n \in \omega \rangle$ of continuous functions on X converging to 0 there is $A \in \mathcal{J}^d$ such that $f_{\mathrm{e}_A(n)} \xrightarrow{\text{QN}} 0$.

$\boxed{\text{QN} \to \text{s}\mathcal{J}\text{QN}}$

$\boxed{\text{s}\mathcal{J}\text{QN} \to \mathcal{J}\text{QN}}$

$\boxed{\text{s}\mathcal{J}\text{QN} \to \text{wQN}}$

A topological space X is a wQN-space if each sequence of continuous real-valued functions converging to zero on X contains a subsequence converging quasi-normally.

Proposition

If $\mathfrak{p} = \mathfrak{c}$ *then for any separable* wQN*-space* X *there is a tall* P*-ideal* $\mathcal{J} \subseteq \mathcal{P}(\omega)$ *such that* X *is an* $s\mathcal{J}$QN*-space.*

Galvin F. and Miller A.W., *γ-sets and other singular sets of real numbers,* Topology Appl. **17** (1984), 145–155.

Bukovský L., Recław I. and Repický M., *Spaces not distinguishing pointwise and quasinormal convergence of real functions,* Topology Appl. **41** (1991), 25–40.

Corollary

If $\mathfrak{p} = \mathfrak{c}$ *there is a tall* P*-ideal* $\mathcal{J} \subseteq \mathcal{P}(\omega)$ *and an* $s\mathcal{J}$QN*-space of cardinality* $\mathfrak{c}$ *which is not a* QN*-space.*

$\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-space

Let $\mathcal{A}$, $\mathcal{B}$ be families of covers of X. A topological space X possesses the property $\mathbf{S_1(\mathcal{A}, \mathcal{B})}$ if for every sequence $\langle \mathcal{U}_n;\ n \in \omega \rangle$ of covers from $\mathcal{A}$ there exist sets $U_n \in \mathcal{U}_n, n \in \omega$ such that $\{U_n;\ n \in \omega\} \in \mathcal{B}$.

A sequence $\langle U_n;\ n \in \omega \rangle$ of subsets of a topological space X is said to be an **$\mathcal{I}$-γ-cover**, if for every n, $U_n \neq X$, and for every $x \in X$, the set $\{n \in \omega : x \notin U_n\}$ belongs to $\mathcal{I}$.

$\mathcal{I}$-Γ - the family of all open $\mathcal{I}$-γ-covers

Γ - the family of all open Fin-γ-covers (i.e., γ-covers)

Haleš J., *On Scheepers' conjecture,* Acta Univ. Carolinae Math. Phys. **46** (2005), 27–31.

Bukovský L. and Haleš J., QN-*spaces,* wQN-*spaces and covering properties,* Topology Appl. **154** (2007), 848–858.

Sakai M., *The sequence selection properties of* $\mathrm{C}_p(X)$, Topology Appl. **154** (2007), 552–560.

$\boxed{\text{any normal QN-space is an } \mathrm{S}_1(\Gamma, \Gamma)\text{-space}}$

Proposition (L. Bukovský – P. Das – J. Š.)

Let $\mathcal{I} \subseteq \mathcal{P}(\omega)$ *be an ideal with a pseudounion* C, $A = \omega \setminus C$. *Then for any* $\mathcal{I}$-γ-*cover* $\langle U_n;\ n \in \omega\rangle$, *the sequence* $\langle U_{e_A(n)};\ n \in \omega\rangle$ *is a* γ-*cover.*

Corollary (L. Bukovský – P. Das – J. Š.)

Let $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(\omega)$ *be ideals with pseudounions. Then any topological space* X *is an* $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-*space if and only if* X *is an* $\mathrm{S}_1(\Gamma, \Gamma)$-*space.*

Proposition (L. Bukovský – P. Das – J. Š.)

Any γ-*set is an* $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-*space.*

Corollary

The implication $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma) \to \mathrm{QN}$ *is not provable.*

A cover $\mathcal{A}$ of X is an ω-cover if for any finite subset F of X there is $A \in \mathcal{A}$ such that $F \subseteq A$.

A topological space X is a γ-set if any open ω-cover of X contains γ-subcover.

$$\lambda(\mathcal{I},\mathcal{J}) = \min\{|\mathcal{A}|;\ \mathcal{A} \subseteq {}^{\omega}\mathcal{I} \wedge (\forall \varphi \in {}^{\omega}\omega)(\exists \langle B_n;\ n \in \omega\rangle \in \mathcal{A})\ \{n;\ \varphi(n) \in B_n\} \in \mathcal{J}^+\}$$

Proposition

Let $\mathcal{I},\mathcal{J} \subseteq \mathcal{P}(\omega)$ be ideals. Then $\aleph_1 \leq \lambda(\mathcal{I},\mathrm{Fin}) \leq \lambda(\mathcal{I},\mathcal{J}) \leq \lambda(\mathrm{Fin},\mathcal{J}) = \mathfrak{b}_{\mathcal{J}} \leq \mathfrak{d}$.

$\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma,\mathcal{J}\text{-}\Gamma)\text{-space})$ denotes the minimal cardinality of a perfectly normal space which is not an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma,\mathcal{J}\text{-}\Gamma)$-space.

Proposition

Let $\mathcal{I},\mathcal{J} \subseteq \mathcal{P}(\omega)$ be ideals. Then non$(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma,\mathcal{J}\text{-}\Gamma)$-space$) = \lambda(\mathcal{I},\mathcal{J})$. In particular, non$(\mathrm{S}_1(\Gamma,\mathcal{J}\text{-}\Gamma)$-space$) = \mathfrak{b}_{\mathcal{J}}$.

Farkas B. and Soukup L., *More on cardinal invariants of analytic P-ideals*, Comment. Math. Univ. Carolin. **50** (2009), 281–295.

Corollary

If $\mathcal{I}$ has a pseudounion and $\mathcal{J}$ is meager then non$(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma,\mathcal{J}\text{-}\Gamma)$-space$) = \mathfrak{b}$. In particular, non$(\mathrm{S}_1(\Gamma,\mathcal{J}\text{-}\Gamma)$-space$) = \mathfrak{b}$ for any analytic $\mathcal{J}$.

Preservation

Proposition

Let $\mathcal{J}_1, \mathcal{J}_2 \subseteq \mathcal{P}(\omega)$ *be ideals such that* $\mathcal{J}_1 \leq_{KB} \mathcal{J}_2$*,* X *being a topological space. If* X *is a* $\mathcal{J}_1\mathrm{QN}$*-space then* X *is a* $\mathcal{J}_2\mathrm{QN}$*-space.*

Proposition

Let $\mathcal{I}_1, \mathcal{I}_2, \mathcal{J}_1, \mathcal{J}_2 \subseteq \mathcal{P}(\omega)$ *be ideals such that* $\mathcal{I}_1 \leq_{K} \mathcal{I}_2$*,* $\mathcal{J}_1 \leq_{KB} \mathcal{J}_2$*,* X *being a topological space. If* X *is an* $\mathrm{S}_1(\mathcal{I}_2\text{-}\Gamma, \mathcal{J}_1\text{-}\Gamma)$*-space then* X *is an* $\mathrm{S}_1(\mathcal{I}_1\text{-}\Gamma, \mathcal{J}_2\text{-}\Gamma)$*-space.*

Corollary

If $\mathcal{I}_1, \mathcal{I}_2, \mathcal{J}_1, \mathcal{J}_2 \subseteq \mathcal{P}(\omega)$ *are ideals such that* $\mathcal{I}_1 \leq_{K} \mathcal{I}_2$*,* $\mathcal{J}_1 \leq_{KB} \mathcal{J}_2$*, then* $\lambda(\mathcal{I}_2, \mathcal{J}_1) \leq \lambda(\mathcal{I}_1, \mathcal{J}_2)$*.*

Thanks for Your attention!