Temperature dependence of sound velocity and hydrodynamics of ultra-relativistic heavy-ion collisions

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International Conference on Strangeness in Quark Matter
24 - 29 June 2007
Motivation

Investigation of the phase transition in quark matter in the RHIC region

- Theoretical calculations describe matter at high temperature region (lattice QCD) or in the low temperatures (hadron gas model).
- Phase transition around the critical temperature $T_C$ with possible first order, cross-over or other type.
- Theoretical exploration of this region in phase diagram is needed.
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Our achievement:

Development of new approach to relativistic hydrodynamics where the temperature-dependent sound velocity is the only thermodynamic input characterizing matter.
Outline

- Thermodynamic relations in Hadron Gas (HG) and Quark-Gluon Plasma (QGP)
  - Temperature dependent sound velocity $c_s(T)$
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- Introduction to Baym et al. method of solving equations of relativistic hydrodynamics
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  ■ Temperature dependent sound velocity $c_s(T)$
■ Introduction to Baym et al. method of solving equations of relativistic hydrodynamics
■ Solutions of the relativistic hydrodynamics
  ■ constant sound velocity $\left( c_s = \frac{1}{\sqrt{3}} \right)$
  ■ $c_s$ based on HG and lattice QCD calculations
  ■ modeling phase transition with temperature dependent sound velocity
Relativistic thermodynamics

classical relativistic ideal gas — hadron gas model

Energy-momentum tensor

\[ T_{\mu\nu}(x) = \int d^3 p \rho p_\mu p_\nu f(x, p) \]

Calculations performed in the rest frame of gas element

\[ u_\mu = (1, 0, 0, 0) \]

Energy density

\[ \epsilon(T, m) = T_{00} = \frac{4}{3} \pi \hbar^3 e \left( \frac{m}{T} \right) \]

Pressure

\[ P(T, m) = T_{33} = \frac{4}{3} \pi \hbar^3 e \left( \frac{m^3}{T^2} \right) \]

Entropy

\[ s(T, m) = s(T, m) + P = \int_{m_{\text{max}}}^{\text{pion}} \int_{m_{\text{max}}}^{\text{nucl}} (M_{\text{Me}} m T s(T, m) + 2 N_{\text{Be}} m T s(T, m)) dm \]

Calculating hadron gas with resonances using Hagedorn mass spectra, i.e.

\[ s(T) = \int_{m_{\text{max}}}^{\text{pion}} \int_{m_{\text{max}}}^{\text{nucl}} (M_{\text{Me}} m T s(T, m) + 2 N_{\text{Be}} m T s(T, m)) dm \]
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\[ T^{\mu\nu}(x) = \int \frac{d^3p}{p^0} p^\mu p^\nu f(x, p) \quad f(p) = \frac{1}{h^3} e^{\frac{p^\alpha u_\alpha}{T}} \]

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\[ T^{\mu\nu}(x) = \int \frac{d^3 p}{p^0} p^\mu p^\nu f(x, p) \quad f(p) = \frac{1}{h^3} e^{\mu - p^\alpha u_\alpha} \]

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**energy density**

\[ \epsilon(T, m) = T^{00} = \frac{4\pi}{h^3} e^{\mu} \left[ 3m^2 T^2 K_2 \left( \frac{m}{T} \right) + m^3 T K_1 \left( \frac{m}{T} \right) \right] \]

**pressure**

\[ P(T, m) = T^{33} = \frac{4\pi}{h^3} e^{\mu} m^2 T^2 K_2 \left( \frac{m}{T} \right) \]

**entropy**

\[ \epsilon + P = sT \quad \Rightarrow \quad s(T, m) = \frac{4\pi}{h^3} e^{\mu} m^3 K_3 \left( \frac{m}{T} \right) \]
Relativistic thermodynamics

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Calculating hadron gas with resonances using Hagedorn mass spectra, i.e. entropy

\[ s(T) = \int_{pion}^{\text{max}} a_M e^{\frac{m}{T_M}} s(T, m) \, dm + 2 \int_{\text{nucI}}^{\text{max}} a_B e^{\frac{m}{T_B}} s(T, m) \, dm \]
Relativistic thermodynamics
Quark-gluon plasma — lattice QCD simulations

- Lattice QCD solutions for pressure

\[ p = c T^4 \sigma \left( \frac{T}{T_C} \right) \]
\[ \sigma (g) = \frac{1 + e^{-a/b}}{1 + e^{(g-a)/b}} \]

Aoki, Fodor, Katz and Szabo:
**JHEP 0601:089,2006**

Relativistic thermodynamics
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Biro and Zimanyi: hep-ph/0607079

- Pressure parametrization
- Entropy
\[
s = \frac{\partial p}{\partial T}
\]

- Energy density
\[
\epsilon = T s - p
\]

- Sound velocity
\[
C_s^2 = \frac{s}{T} \frac{1}{\frac{\partial s}{\partial T}}
\]

Aoki, Fodor, Katz and Szabo:

JHEP 0601:089, 2006

Mikołaj Chojnacki (IFJ PAN)
Thermodynamical functions

entropy $s$, pressure $p$, energy density $\epsilon$ and sound velocity $c_s$

![Graphs showing thermodynamical functions for Hadron Gas, Pion Gas, and lattice QCD. The graphs compare the ratios $P/T^4$, $\epsilon/T^4$, $s/T^3$, and $c_s^2$ against temperature $T$ in GeV.]
Relativistic hydrodynamics

Space time evolution in ultra-relativistic heavy-ion collisions
Equations of relativistic hydrodynamics

- Energy and momentum conservation law

\[ \partial_\mu T^{\mu\nu} = 0 \]
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\[ \partial_{\mu} T^{\mu\nu} = 0 \]

- Energy-momentum tensor of the perfect fluid

\[ T^{\mu\nu} = (\epsilon + P) u^{\mu} u^{\nu} - P g^{\mu\nu} \]

\( \epsilon \) - energy density, \( P \) - pressure, \( u^{\mu} \) - four-velocity of the fluid element
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- At mid-rapidity (\( y = 0 \)) for RHIC energies

\[ \mu_B = 0 \]

Temperature is the only thermodynamic parameter
System Geometry

- Cylindrical coordinates \((r, \phi)\)

\[
\begin{align*}
  r &= \sqrt{x^2 + y^2} \\
  \phi &= \arctan \frac{y}{x} \\
  v &= \sqrt{v_T^2 + v_r^2} \\
  \alpha &= \arctan \frac{v_T}{v_r}
\end{align*}
\]
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  \]

- **Boost – invariant symmetry**

  Values of physical quantities at \(z \neq 0\) may be calculated by Lorentz transformation

  Lorentz factor \(\gamma = \left(1 - v^2\right)^{\frac{1}{2}}\)
Equations in cylindrical coordinates

- Equations in covariant form

\[ u^\mu \partial_\mu (T u^\nu) = \partial^\nu T \]
\[ \partial_\mu (s u^\mu) = 0 \]
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- Non-covariant notation (cylindrical symmetry)

Baym, Friman, Blaizot, Soyeur, Czyz

\[ v_r \frac{\partial \ln T}{\partial t} + \frac{\partial \ln T}{\partial r} + \frac{\partial \theta}{\partial t} + v_r \frac{\partial \theta}{\partial r} = 0 \]
\[ \frac{\partial \ln s}{\partial t} + v_r \frac{\partial \ln s}{\partial r} + v_r \frac{\partial \theta}{\partial t} + \frac{\partial \theta}{\partial r} + \frac{1}{t} + \frac{v_r}{r} = 0 \]

where \( \theta = \tanh^{-1} v_r \) is the transverse rapidity
Temperature dependent sound velocity $c_s(T)$

- Relation between $T$ and $s$ needed to close the set of three equations

\[
c_s^2(T) = \frac{\partial P}{\partial \epsilon} = \frac{s}{T} \frac{\partial T}{\partial s}
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Temperature dependent sound velocity $c_s(T)$

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- Potential $\Phi$

\[
d\Phi = \frac{1}{c_s} \, d \ln T = c_s \, d \ln s
\]

- Potential $\Phi$ dependent on $T$

\[
\Phi_T(T) = \int_{T_0}^{T} \frac{d \ln T'}{c_s(T')} + \Phi(T_0)
\]

- Entropy as a function of $T$

\[
s_T(T) = s(T_0) \exp \left[ \int_{T_0}^{T} \frac{d \ln T'}{c_s^2(T')} \right]
\]
Temperature dependent sound velocity $c_s(T)$

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$$s_T(T) = s(T_0) \exp \left[ \int_{T_0}^T \frac{d\ln T'}{c_s^2(T')} \right]$$

- Condition against shock wave formation

$$\frac{\partial}{\partial T} \left( \frac{s c_s}{T} \right) = 1 - c_s^2 + c_s T \frac{\partial c_s}{\partial T} \geq 0$$
Characteristic form of hydrodynamical equations
with cylindrical symmetry and boost-invariance. 1 + 1 in transverse direction.

auxiliary functions $A_{\pm} = \Phi \pm \theta$

$$\frac{\partial}{\partial t} A_{\pm} (r, t) + \frac{v_r \pm c_s}{1 \pm v_r c_s} \frac{\partial}{\partial r} A_{\pm} (r, t) + \frac{c_s}{1 \pm v_r c_s} \left( \frac{v_r}{r} + \frac{1}{t} \right) = 0$$
Characteristic form of hydrodynamical equations

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\frac{\partial}{\partial t} A_\pm (r, t) + \frac{v_r \pm c_s}{1 \pm v_r c_s} \frac{\partial}{\partial r} A_\pm (r, t) + \frac{c_s}{1 \pm v_r c_s} \left( \frac{v_r}{r} + \frac{1}{t} \right) = 0
$$

- velocity $\quad v_r = \tanh \left( \frac{A_+ - A_-}{2} \right)$
- potential $\Phi \quad \Phi = \frac{A_+ + A_-}{2}$
- temperature $\quad T = T_\Phi \left[ \frac{A_+ + A_-}{2} \right]$
- sound velocity $\quad c_s = c_s \left( T_\Phi \left[ \frac{A_+ + A_-}{2} \right] \right)$
Boundary conditions

- Single function $A$ to describe $A_{\pm}$

\[
A_+ (r, t) = A(r, t) \quad r > 0
\]
\[
A_- (r, t) = A(-r, t) \quad r < 0
\]
Boundary conditions

- Single function $A$ to describe $A_{\pm}$

$$
A_+ (r, t) = A(r, t) \quad r > 0 \\
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- Automatically fulfilled boundary conditions at $r = 0$

$$
\nu (r = 0, t) = 0 \\
\frac{\partial T}{\partial r} \mid_{r=0} = 0
$$
Boundary conditions

- Single function $A$ to describe $A_{\pm}$
  
  $A_{+}(r, t) = A(r, t) \quad r > 0$
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- Automatically fulfilled boundary conditions at $r = 0$
  
  $v(r = 0, t) = 0 \quad \frac{\partial T}{\partial r} |_{r=0} = 0$

Characteristic form of 1+1 hydro equation

$$\frac{\partial}{\partial t} A(r, t) + \frac{v_{r \pm} c_{s}}{1 \pm v_{r} c_{s}} \frac{\partial}{\partial r} A(r, t) + \frac{c_{s}}{1 \pm v_{r} c_{s}} \left( \frac{v_{r}}{r} + \frac{1}{t} \right) = 0$$
Initial condition

- Initial entropy density $s$ is proportional to the density of wounded nucleons $dN_w/dr$.

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$$T (r, t = t_0) = T_s \left( s_0 \frac{dN_w}{dr} / \frac{dN_w(0)}{dr} \right)$$

$$\frac{dN_w}{dr} = T_A (r) = 2 \int_0^\infty dz \frac{\rho_0}{1 + e^{\frac{r-r_0}{a}}}$$

- $A = 197$, $\rho_0 = 0.17 \text{ fm}^{-3}$
- $r_0 = 6.37 \text{ fm}$, $a = 0.54 \text{ fm}$
Initial condition

- Hubble-like transverse flow

\[ v_0(r) = v(r, t = t_0) = \frac{H_0 r}{\sqrt{1 + H_0^2 r^2}} \]

- Initial entropy density \( s \) is proportional to the density of wounded nucleons \( dN_w/dr \)

\[ T(r, t = t_0) = T_s \left( s_0 \frac{dN_w}{dr} / \frac{dN_w(0)}{dr} \right) \]

\[ \frac{dN_w}{dr} = T_A(r) = 2 \int_0^\infty dz \frac{\rho_0}{1 + e^{\frac{r-r_0}{a}}} \]

\[ A = 197, \quad \rho_0 = 0.17 \text{ fm}^{-3} \]

\[ r_0 = 6.37 \text{ fm}, \quad a = 0.54 \text{ fm} \]
Results

- starting time of hydrodynamic evolution
  \[ t_0 = 1.0 \text{[fm/c]} \]

- initial central temperature
  \[ T_0 = 2T_C = 340 \text{[MeV]} \]

- no initial transverse flow
  \[ H_0 = 0 \]
Constant sound velocity $c_s^2 = \frac{1}{3}$

graphs showing $c_s^2(r)$, $v(r)$, $T(r)$, and $T/t$ as functions of $r$ for different times.
Modeling of the sound velocity at $T_C$

\[ \frac{s}{T^3} \]

\[ \frac{\epsilon}{T^4} \]

\[ \frac{dp}{d\epsilon} = (c_S^2) \]

\[ 1 - c_s^2 + T \frac{dc_s}{dT} \]

\[ T/T_c \]

\[ T_3, T_4 \]

\[ \epsilon_0, \epsilon_1 \]

\[ P \text{ [GeV/fm}^3\text{]} \]

\[ \text{MeV} \]

\[ \text{GeV} \]

\[ \text{fm}^3 \]
$c_s$ case I — simplest interpolation between HG and lattice QCD
$c_s$ case II — $c_s$ drops by 25% at $T_C$
$c_s$ case III — $c_s$ drops by 50% at $T_C$
Summary

- We have developed a new and efficient method of solving relativistic hydrodynamics, reformulating the appropriate partial differential equations in such a way that the sound velocity in medium is the only input characterizing the properties of matter.
- We introduced different parameterisations of the sound velocity in the region of the phase transition.
- We showed that only the system with the case I EOS, with a moderate minimum at critical temperature, is able experience short lifetime, compatible with the recent estimates based on the HBT interferometry.