The Three-Loop Splitting Functions in QCD

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with

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Introduction

Notation, Method ($\rightarrow$ parallel session)

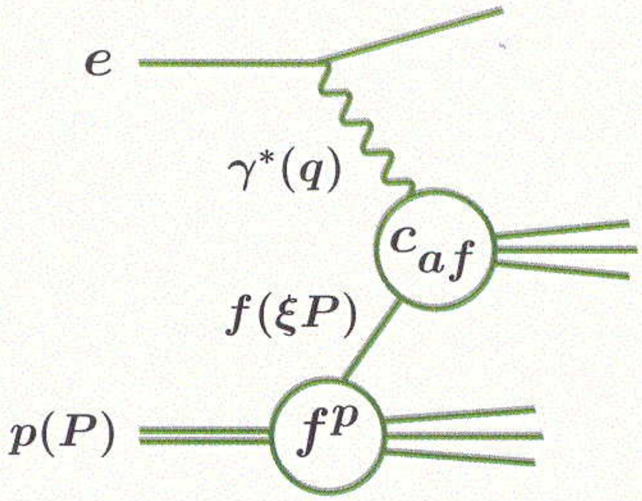
Non-singlet and singlet results

Large x, small x, numerical effects

Summary

hep-ph/0403192 (ns), hep-ph/0404111 (s)

DIS in perturbative QCD



Kinematic variables

$$Q^2 = -q^2$$
$$x = Q^2/(2P\cdot q)$$

Structure functions F_a [up to $\mathcal{O}(1/Q^2)$]

$$\eta_a F_a^p(x,Q^2) = \sum_f \left[c_{a,f}(a_S) \otimes f^p(\mu^2)\right](\xi)$$

Parton distribution functions f (PDF's)

$$\frac{d}{d\ln\mu^2} f(\xi,\mu^2) = \sum_{f'} \left[P_{ff'}(a_S) \otimes f'(\mu^2)\right](\xi)$$

Coefficient functions c, splitting functions P

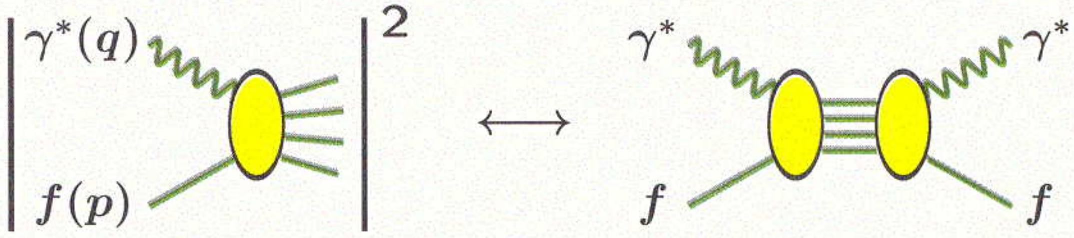
$$P = a_S P^{(0)} + a_S^2 P^{(1)} + a_S^3 P^{(2)} + \ldots$$
$$c = \underbrace{a_S^{n_a}\left[c^{(0)} + a_S\, c^{(1)}\right.}_{\text{NLO:}} \left. + a_S^2\, c^{(2)} + \ldots\right]$$

NLO:
Standard approximation for pert. QCD analyses

Next-to-next-to-leading order (NNLO): $c^{(2)}$, $P^{(2)}$

How the calculation is done

Optical theorem: forward Compton amplitude



Coefficient of $(2p\cdot q)^N$ $\leftrightarrow$ N-th Mellin moment

$$A^N = \int_0^1 dx\, x^{N-1} A(x)$$

$P_{\rm gg}$, $P_{\rm gq}$: DIS with scalar ϕ coupling to $G^a_{\mu\nu}G^{\mu\nu}_a$

Gluon polarization sum $\leftrightarrow$ diagrams with ghost h

	tree	1-loop	2-loop	3-loop
qγ	1	3	25	359
gγ		2	17	345
hγ			2	56
qW	1	3	32	589
qϕ		1	23	696
gϕ	1	8	218	6378
hϕ		1	33	1184
sum	3	18	350	9607

Highly optimised symbolic treatment needed

FORM J. Vermaseren (1989-2004)

Capabilities substantially extended for this project

Evolution of parton distributions

Non-singlet and singlet distributions $q^{\pm}$, q^{v} and q_S, g

$$
\begin{aligned}
q_{ik}^{\pm} &= q_i \pm \bar{q}_i - (q_k \pm \bar{q}_k) \\
q^{v} &= \sum_{r=1}^{n_f} (q_r - \bar{q}_r) \\
f_S &= \begin{pmatrix} q_S \\ g \end{pmatrix} \;,\; q_S = \sum_{r=1}^{n_f} (q_r + \bar{q}_r)
\end{aligned}
$$

Evolution equations for $\mu_r = \mu_f$

$$
\frac{d}{d \ln \mu_f^2} f(x, \mu_f^2) = \Big[P(\alpha_s(\mu_f^2)) \otimes f(\mu_f^2) \Big] (x)
$$

$2n_f - 1$ scalar (ns) equations $+$ 2×2 singlet case

Mellin convolution

$$
[a \otimes b](x) = \int_x^1 d\xi \, a\left(\frac{x}{\xi}\right) b(\xi)
$$

Splitting-function combinations

$$
\begin{aligned}
P_{\rm ns}^{\pm} \quad &, \quad P_{\rm ns}^{v} = P_{\rm ns}^{-} + P_{\rm ns}^{S} \\
P_{\rm S} &= \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \\
P_{qq} &= P_{\rm ns}^{+} + P_{\rm ps}
\end{aligned}
$$

LO: $P_{\rm ns}^{+} = P_{\rm ns}^{-} = P_{\rm ns}^{v} = P_{qq}$, NLO: $P_{\rm ns}^{-} = P_{\rm ns}^{v}$

Basic functions in N- and x-space

N-space: harmonic sums

$$S_{\pm m}(M) = \sum_{i=1}^{M} \frac{(\pm 1)^i}{i^m}$$

$$S_{\pm m_1, m_2, \ldots, m_k}(M) = \sum_{i=1}^{M} \frac{(\pm 1)^i}{i^{m_1}} S_{m_2, \ldots, m_k}(i)$$

Weight = sum of absolute values of indices

Notation: $\mathrm{N_{\pm i}}\, S_{\vec{m}} = S_{\vec{m}}(N \pm i)$

Vermaseren (98)

x-space: harmonic polylogarithms

$$H_0(x) = \ln x \;, \qquad H_{\pm 1}(x) = \mp \ln(1 \mp x)$$

$$H_{m_1, \ldots, m_w}(x) = \begin{cases} \dfrac{1}{w!} \ln^w x \;, & \vec{m} = \vec{0} \\ \displaystyle\int_0^x dz\, f_{m_1}(z)\, H_{m_2, \ldots, m_w}(z) \;, & \text{else} \end{cases}$$

with

$$f_0(x) = \frac{1}{x} \;, \qquad f_{\pm 1}(x) = \frac{1}{1 \mp x}$$

Weight = number of indices $(0, \pm 1)$

$$H_{\underbrace{0, \ldots, 0}_{m}, \pm 1, \underbrace{0, \ldots, 0}_{n}, \pm 1, \ldots}(x) \equiv H_{\pm(m+1), \pm(n+1), \ldots}(x)$$

Remiddi, Vermaseren (99)

$$P_{\mathrm{gg}}^{(0)}(x) =$$

$$C_A\left(4[(1-x)^{-1}+x^{-1}-2+x-x^2]+\frac{11}{3}\delta(1-x)\right)-\frac{2}{3}n_f\,\delta(1-x)$$

$$P_{\mathrm{gg}}^{(1)}(x) =$$

$$\begin{aligned}
&4C_An_f\left(1-x-\frac{10}{9}p_{gg}(x)-\frac{13}{9}\left(\frac{1}{x}-x^2\right)-\frac{2}{3}(1+x)\mathrm{H}_0-\frac{2}{3}\delta(1-x)\right)+4C_A{}^2\Big(27\\
&+(1+x)\left[\frac{11}{3}\mathrm{H}_0+8\mathrm{H}_{0,0}-\frac{27}{2}\right]+2p_{gg}(-x)\left[\mathrm{H}_{0,0}-2\mathrm{H}_{-1,0}-\zeta_2\right]-\frac{67}{9}\left(\frac{1}{x}-x^2\right)\\
&-12\mathrm{H}_0-\frac{44}{3}x^2\mathrm{H}_0+2p_{gg}(x)\left[\frac{67}{18}-\zeta_2+\mathrm{H}_{0,0}+2\mathrm{H}_{1,0}+2\mathrm{H}_2\right]+\delta(1-x)\left[\frac{8}{3}+3\zeta_3\right]\Big)\\
&+4C_Fn_f\left(2\mathrm{H}_0+\frac{2}{3}\frac{1}{x}+\frac{10}{3}x^2-12+(1+x)\left[4-5\mathrm{H}_0-2\mathrm{H}_{0,0}\right]-\frac{1}{2}\delta(1-x)\right)
\end{aligned}$$

with

$$p_{gg}(x)=(1-x)^{-1}+x^{-1}-2+x-x^2$$

$$P_{\mathrm{gg}}^{(2)}(x) =$$

$$16C_AC_Fn_f\Big(x^2\Big[\frac{4}{9}\mathrm{H}_2+3\mathrm{H}_{1,0}-\frac{97}{12}\mathrm{H}_1+\frac{8}{3}\mathrm{H}_{-2,0}-\frac{2}{3}\mathrm{H}_0\zeta_2+\frac{103}{27}\mathrm{H}_0-\frac{16}{3}\zeta_2+2\mathrm{H}_3$$

$$-6\mathrm{H}_{-1,0}+2\mathrm{H}_{2,0}+\frac{127}{18}\mathrm{H}_{0,0}-\frac{511}{12}\Big]+p_{gg}(x)\Big[2\zeta_3-\frac{55}{24}\Big]+\frac{4}{3}(\frac{1}{x}-x^2)\Big[\frac{17}{24}\mathrm{H}_{1,0}-\frac{43}{18}\mathrm{H}_0$$

$$-\frac{521}{144}\mathrm{H}_1-\frac{6923}{432}-\frac{1}{2}\mathrm{H}_{2,1}+2\mathrm{H}_1\zeta_2+\mathrm{H}_0\zeta_2-2\mathrm{H}_{1,0,0}+\frac{1}{12}\mathrm{H}_{1,1}-\mathrm{H}_{1,1,0}-\mathrm{H}_{1,1,1}\Big]-\frac{175}{12}\mathrm{H}_2$$

$$+6\mathrm{H}_{-1,0}+8\mathrm{H}_0\zeta_3-6\mathrm{H}_{-2,0}-\frac{53}{6}\mathrm{H}_0\zeta_2-\frac{49}{2}\mathrm{H}_0+\frac{185}{4}\zeta_2+\frac{511}{12}-\frac{1}{2}\mathrm{H}_{2,0}-3\mathrm{H}_{1,0}-4\mathrm{H}_{0,0,0,0}$$

$$-\frac{67}{12}\mathrm{H}_{0,0}+\frac{43}{2}\zeta_3-\mathrm{H}_{2,1}+\frac{97}{12}\mathrm{H}_1-4\zeta_2{}^2-\frac{9}{2}\mathrm{H}_3-8\mathrm{H}_{-3,0}+\frac{33}{2}\mathrm{H}_{0,0,0}+\frac{4}{3}(\frac{1}{x}+x^2)\Big[\frac{1}{2}\mathrm{H}_2-\mathrm{H}_{2,0}$$

$$+\frac{11}{3}\mathrm{H}_{-1,0}+\mathrm{H}_{-2,0}+\frac{19}{6}\zeta_2+2\zeta_3-\mathrm{H}_{-1}\zeta_2-4\mathrm{H}_{-1,-1,0}-\frac{1}{2}\mathrm{H}_{-1,0,0}-\mathrm{H}_{-1,2}\Big]+(1-x)\Big[9\mathrm{H}_1\zeta_2$$

$$+12\mathrm{H}_{0,0,0,0}-\frac{293}{108}+\frac{61}{6}\mathrm{H}_0\zeta_2-\frac{7}{3}\mathrm{H}_{1,0}-\frac{857}{36}\mathrm{H}_1-9\mathrm{H}_0\zeta_3+16\mathrm{H}_{-2,-1,0}-4\mathrm{H}_{-2,0,0}+8\mathrm{H}_{-2}\zeta_2$$

$$-\frac{13}{2}\mathrm{H}_{1,0,0}+\frac{3}{4}\mathrm{H}_{1,1}-\mathrm{H}_{1,1,0}-\mathrm{H}_{1,1,1}\Big]+(1+x)\Big[\frac{1}{6}\mathrm{H}_{2,0}-\frac{95}{3}\mathrm{H}_{-1,0}-\frac{149}{36}\mathrm{H}_2+\frac{3451}{108}\mathrm{H}_0$$

$$-7\mathrm{H}_{-2,0}+\frac{302}{9}\mathrm{H}_{0,0}+\frac{19}{6}\mathrm{H}_3-\frac{991}{36}\zeta_2-\frac{163}{6}\zeta_3-\frac{35}{3}\mathrm{H}_{0,0,0}+\frac{17}{6}\mathrm{H}_{2,1}-\frac{43}{10}\zeta_2{}^2+13\mathrm{H}_{-1}\zeta_2$$

$$+18\mathrm{H}_{-1,-1,0}-\mathrm{H}_{3,1}-6\mathrm{H}_4-4\mathrm{H}_{-1,2}+6\mathrm{H}_{0,0}\zeta_2+8\mathrm{H}_2\zeta_2-7\mathrm{H}_{2,0,0}-2\mathrm{H}_{2,1,0}-2\mathrm{H}_{2,1,1}-4\mathrm{H}_{3,0}$$

$$-9\mathrm{H}_{-1,0,0}\Big]-\frac{241}{288}\delta(1-x)\Big)+16C_An_f{}^2\Big(\frac{19}{54}\mathrm{H}_0-\frac{1}{24}x\mathrm{H}_0-\frac{1}{27}p_{gg}(x)+\frac{13}{54}(\frac{1}{x}-x^2)\Big[\frac{5}{3}-\mathrm{H}_1\Big]$$

$$+(1-x)\Big[\frac{11}{72}\mathrm{H}_1-\frac{71}{216}\Big]+\frac{2}{9}(1+x)\Big[\zeta_2+\frac{13}{12}x\mathrm{H}_0-\frac{1}{2}\mathrm{H}_{0,0}-\mathrm{H}_2\Big]+\frac{29}{288}\delta(1-x)\Big)$$

$$+16C_A{}^2n_f\Big(x^2\Big[\zeta_3+\frac{11}{9}\zeta_2+\frac{11}{9}\mathrm{H}_{0,0}-\frac{2}{3}\mathrm{H}_3+\frac{2}{3}\mathrm{H}_0\zeta_2+\frac{1639}{108}\mathrm{H}_0-2\mathrm{H}_{-2,0}\Big]+\frac{1}{3}p_{gg}(x)\Big[\frac{10}{3}\zeta_2$$

$$-\frac{209}{36}-8\zeta_3-2\mathrm{H}_{-2,0}-\frac{1}{2}\mathrm{H}_0-\frac{10}{3}\mathrm{H}_{0,0}-\frac{20}{3}\mathrm{H}_{1,0}-\mathrm{H}_{1,0,0}-\frac{20}{3}\mathrm{H}_2-\mathrm{H}_3\Big]+\frac{10}{9}p_{gg}(-x)\Big[\zeta_2$$

$$+2\mathrm{H}_{-1,0}+\frac{3}{10}\mathrm{H}_0\zeta_2-\mathrm{H}_{0,0}\Big]+\frac{1}{3}(\frac{1}{x}-x^2)\Big[\mathrm{H}_3-\mathrm{H}_0\zeta_2-\frac{13}{3}\mathrm{H}_2+\frac{5443}{108}-3\mathrm{H}_1\zeta_2+\frac{205}{36}\mathrm{H}_1$$

$$-\frac{13}{3}\mathrm{H}_{1,0}+\mathrm{H}_{1,0,0}\Big]+(\frac{1}{x}+x^2)\Big[\frac{151}{54}\mathrm{H}_0-\frac{8}{3}\zeta_2+\frac{1}{3}\mathrm{H}_{-1}\zeta_2-\zeta_3+2\mathrm{H}_{-1,-1,0}-\frac{2}{3}\mathrm{H}_{-1,0,0}$$

$$-\frac{37}{9}\mathrm{H}_{-1,0}+\frac{2}{3}\mathrm{H}_{-1,2}\Big]+(1-x)\Big[\frac{5}{6}\mathrm{H}_{-2,0}+\mathrm{H}_{-3,0}+2\mathrm{H}_{0,0,0}-\frac{269}{36}\zeta_2-\frac{4097}{216}-3\mathrm{H}_{-2}\zeta_2$$

$$-6\mathrm{H}_{-2,-1,0}+3\mathrm{H}_{-2,0,0}-\frac{7}{2}\mathrm{H}_1\zeta_2+\frac{677}{72}\mathrm{H}_1+\mathrm{H}_{1,0}+\frac{7}{4}\mathrm{H}_{1,0,0}\Big]+(1+x)\Big[\frac{193}{36}\mathrm{H}_2-\frac{11}{2}\mathrm{H}_{-1}\zeta_2$$

$$+\frac{39}{20}\zeta_2{}^2-\frac{7}{12}\mathrm{H}_3-\frac{53}{9}\mathrm{H}_{0,0}+\frac{7}{12}\mathrm{H}_0\zeta_2-\frac{5}{2}\mathrm{H}_{0,0}\zeta_2+5\zeta_3-7\mathrm{H}_{-1,-1,0}+\frac{77}{6}\mathrm{H}_{-1,0}+\frac{9}{2}\mathrm{H}_{-1,0,0}$$

$$+2\mathrm{H}_{-1,2}-3\mathrm{H}_2\zeta_2-\frac{2}{3}\mathrm{H}_{2,0}+\frac{3}{2}\mathrm{H}_{2,0,0}+\frac{3}{2}\mathrm{H}_4\Big]+\frac{1}{9}\zeta_2+7\mathrm{H}_{-2,0}+2\mathrm{H}_2+\frac{458}{27}\mathrm{H}_0+\mathrm{H}_{0,0}\zeta_2$$

$$+\frac{3}{2}\zeta_2{}^2+4\mathrm{H}_{-3,0}-x\Big[\frac{131}{12}\mathrm{H}_{0,0}-\frac{8}{3}\mathrm{H}_0\zeta_2+\frac{7}{2}\mathrm{H}_3-\mathrm{H}_{0,0,0,0}+\frac{7}{6}\mathrm{H}_{0,0,0}+\frac{1943}{216}\mathrm{H}_0+6\mathrm{H}_0\zeta_3\Big]$$

$$-\delta(1-x)\Big[\frac{233}{288}+\frac{1}{6}\zeta_2+\frac{1}{12}\zeta_2{}^2+\frac{5}{3}\zeta_3\Big]\Big)+16C_A{}^3\Big(x^2\Big[33\mathrm{H}_{-2,0}+33\mathrm{H}_0\zeta_2-\frac{1249}{18}\mathrm{H}_{0,0}$$

$$-44\mathrm{H}_{0,0,0}-\frac{110}{3}\mathrm{H}_3-\frac{44}{3}\mathrm{H}_{2,0}+\frac{85}{6}\zeta_2+\frac{6409}{108}\mathrm{H}_0\Big]+p_{gg}(x)\Big[\frac{245}{24}-\frac{67}{9}\zeta_2-\frac{3}{10}\zeta_2{}^2+\frac{11}{3}\zeta_3$$

$$-4\mathrm{H}_{-3,0}+6\mathrm{H}_{-2}\zeta_2+4\mathrm{H}_{-2,-1,0}+\frac{11}{3}\mathrm{H}_{-2,0}-4\mathrm{H}_{-2,0,0}-4\mathrm{H}_{-2,2}+\frac{1}{6}\mathrm{H}_0-7\mathrm{H}_0\zeta_3+\frac{67}{9}\mathrm{H}_{0,0}$$

$$-8\mathrm{H}_{0,0}\zeta_2+4\mathrm{H}_{0,0,0,0}-6\mathrm{H}_1\zeta_3-4\mathrm{H}_{1,-2,0}+10\mathrm{H}_{2,0,0}-6\mathrm{H}_{1,0}\zeta_2+8\mathrm{H}_{1,0,0,0}+8\mathrm{H}_{1,1,0,0}+8\mathrm{H}_4$$

$$+\frac{134}{9}\mathrm{H}_{1,0}+\frac{11}{6}\mathrm{H}_{1,0,0}+8\mathrm{H}_{1,2,0}+8\mathrm{H}_{1,3}+\frac{134}{9}\mathrm{H}_2-4\mathrm{H}_2\zeta_2+8\mathrm{H}_{3,1}+8\mathrm{H}_{2,2}+\frac{11}{6}\mathrm{H}_3+10\mathrm{H}_{3,0}$$

$$+8\mathrm{H}_{2,1,0}\Big]+p_{gg}(-x)\Big[\frac{11}{2}\zeta_2{}^2-\frac{11}{6}\mathrm{H}_0\zeta_2-4\mathrm{H}_{-3,0}+16\mathrm{H}_{-2}\zeta_2-12\mathrm{H}_{-2,2}-\frac{134}{9}\mathrm{H}_{-1,0}+2\mathrm{H}_2\zeta_2$$

$$+8\mathrm{H}_{-2,-1,0}+12\mathrm{H}_{-1}\zeta_3-18\mathrm{H}_{-2,0,0}+8\mathrm{H}_{-1,-2,0}-16\mathrm{H}_{-1,-1}\zeta_2+24\mathrm{H}_{-1,-1,0,0}+16\mathrm{H}_{-1,-1,2}$$

$$+18\mathrm{H}_{-1,0}\zeta_2-16\mathrm{H}_{-1,0,0,0}-4\mathrm{H}_{-1,2,0}-16\mathrm{H}_{-1,3}-5\mathrm{H}_0\zeta_3-8\mathrm{H}_{0,0}\zeta_2+4\mathrm{H}_{0,0,0,0}+2\mathrm{H}_{3,0}$$

$$-\frac{67}{9}\zeta_2+\frac{67}{9}\mathrm{H}_{0,0}+8\mathrm{H}_4\Big]+\Big(\frac{1}{x}-x^2\Big)\Big[\frac{16619}{162}+\frac{22}{3}\mathrm{H}_{2,0}-\frac{55}{2}\zeta_3-\frac{11}{2}\mathrm{H}_0\zeta_2-\frac{67}{9}\mathrm{H}_2-\frac{67}{9}\mathrm{H}_{1,0}$$

$$-\frac{413}{108}\mathrm{H}_1-\frac{11}{2}\mathrm{H}_1\zeta_2+\frac{33}{2}\mathrm{H}_{1,0,0}\Big]+11(\frac{1}{x}+x^2)\Big[\frac{71}{54}\mathrm{H}_0-\frac{1}{6}\mathrm{H}_3-\frac{389}{198}\zeta_2-\frac{2}{3}\mathrm{H}_{-2,0}-\frac{1}{2}\mathrm{H}_{-1}\zeta_2$$

$$+\mathrm{H}_{-1,-1,0}-\frac{523}{198}\mathrm{H}_{-1,0}+\frac{8}{3}\mathrm{H}_{-1,0,0}+\mathrm{H}_{-1,2}\Big]+(1-x)\Big[\frac{31}{36}\mathrm{H}_1+\frac{27}{2}\mathrm{H}_{1,0}-\frac{25}{2}\mathrm{H}_{1,0,0}-4\mathrm{H}_{-3,0}$$

$$-\frac{263}{12}\mathrm{H}_{0,0}-\frac{29}{3}\mathrm{H}_{0,0,0}-\frac{19}{3}\mathrm{H}_{-2,0}-\frac{11317}{108}-4\mathrm{H}_{-2}\zeta_2-8\mathrm{H}_{-2,-1,0}-12\mathrm{H}_{-2,0,0}-\frac{3}{2}\mathrm{H}_1\zeta_2\Big]$$

$$+(1+x)\Big[\frac{27}{2}\mathrm{H}_0\zeta_2-\frac{43}{6}\mathrm{H}_3+\frac{29}{3}\mathrm{H}_{2,0}+\frac{4651}{216}\mathrm{H}_0-\frac{329}{18}\zeta_2+\frac{11}{2}(1+x)\zeta_3-\frac{43}{5}\zeta_2{}^2-\frac{215}{6}\mathrm{H}_{-1,0}$$

$$-22\mathrm{H}_{0,0}\zeta_2-8\mathrm{H}_0\zeta_3-3\mathrm{H}_{-1,-1,0}+38\mathrm{H}_{-1,0,0}+25\mathrm{H}_{-1,2}+10\mathrm{H}_{2,0,0}-4\mathrm{H}_2\zeta_2+16\mathrm{H}_{3,0}+26\mathrm{H}_4$$

$$-\frac{158}{9}\mathrm{H}_2-\frac{53}{2}\mathrm{H}_{-1}\zeta_2\Big]-29\mathrm{H}_{0,0}-\frac{40}{3}\mathrm{H}_{0,0,0}+27\mathrm{H}_{-2,0}+\frac{41}{3}\mathrm{H}_0\zeta_2-20\mathrm{H}_3-24\mathrm{H}_{2,0}+\frac{53}{6}\zeta_2$$

$$+\frac{601}{12}\mathrm{H}_0+24\zeta_3+2\zeta_2{}^2+27\mathrm{H}_2-4\mathrm{H}_{0,0}\zeta_2-16\mathrm{H}_0\zeta_3-16\mathrm{H}_{-3,0}+28x\mathrm{H}_{0,0,0,0}+\delta(1-x)\Big[\frac{79}{32}$$

$$-\zeta_2\zeta_3+\frac{1}{6}\zeta_2+\frac{11}{24}\zeta_2{}^2+\frac{67}{6}\zeta_3-5\zeta_5\Big]\Big)+16C_Fn_f{}^2\Big(\frac{2}{9}x^2\Big[\frac{11}{6}\mathrm{H}_0+\mathrm{H}_2-\zeta_2+2\mathrm{H}_{0,0}-9\Big]+\frac{1}{3}\mathrm{H}_2$$

$$-\frac{1}{3}\zeta_2-\frac{10}{3}\mathrm{H}_0-\frac{1}{3}\mathrm{H}_{0,0}+2+\frac{2}{9}(\frac{1}{x}-x^2)\Big[\frac{8}{3}\mathrm{H}_1-2\mathrm{H}_{1,0}-\mathrm{H}_{1,1}-\frac{77}{18}\Big]-(1-x)\Big[\frac{1}{3}\mathrm{H}_{1,0}+\frac{1}{6}\mathrm{H}_{1,1}$$

$$+\frac{4}{9}+\frac{13}{6}\mathrm{H}_1+x\mathrm{H}_1\Big]+\frac{1}{3}(1+x)\Big[\frac{68}{9}\mathrm{H}_0-\frac{4}{3}\mathrm{H}_2+\frac{4}{3}\zeta_2+\frac{29}{6}\mathrm{H}_{0,0}-\zeta_3+2\mathrm{H}_0\zeta_2-\mathrm{H}_{0,0,0}-2\mathrm{H}_3$$

$$-\mathrm{H}_{2,1}-2\mathrm{H}_{2,0}\Big]+\frac{11}{144}\delta(1-x)\Big)+16C_F{}^2n_f\Big(\frac{4}{3}x^2\Big[\frac{163}{16}+\frac{27}{8}\mathrm{H}_0+\frac{7}{2}\mathrm{H}_{0,0}-\mathrm{H}_{2,0}-\zeta_2+\frac{9}{4}\mathrm{H}_{1,0}$$

$$-\mathrm{H}_{2,1}+\frac{1}{2}\mathrm{H}_{0,0,0}+\frac{85}{16}\mathrm{H}_1+\mathrm{H}_2-2\mathrm{H}_{-2,0}-\frac{3}{2}\zeta_3\Big]+\frac{4}{3}(\frac{1}{x}-x^2)\Big[\frac{31}{16}\mathrm{H}_1-\frac{11}{16}-\frac{5}{4}\mathrm{H}_{1,0}+\frac{1}{2}\mathrm{H}_{1,0,0}$$

$$-\mathrm{H}_1\zeta_2-\mathrm{H}_{1,1}+\mathrm{H}_{1,1,0}+\mathrm{H}_{1,1,1}+\zeta_3\Big]+\frac{4}{3}(\frac{1}{x}+x^2)\Big[\mathrm{H}_{-1}\zeta_2+2\mathrm{H}_{-1,-1,0}-\mathrm{H}_{-1,0,0}\Big]+\frac{215}{12}\mathrm{H}_{0,0}$$

$$+\frac{20}{3}\mathrm{H}_0-\frac{131}{6}+3\mathrm{H}_{2,0}+\frac{205}{12}x\zeta_2-3\mathrm{H}_{1,0}+\mathrm{H}_{2,1}-\frac{85}{12}\mathrm{H}_1+\frac{11}{4}\mathrm{H}_2+8\mathrm{H}_{-2,0}+2\zeta_2{}^2-\mathrm{H}_0\zeta_2$$

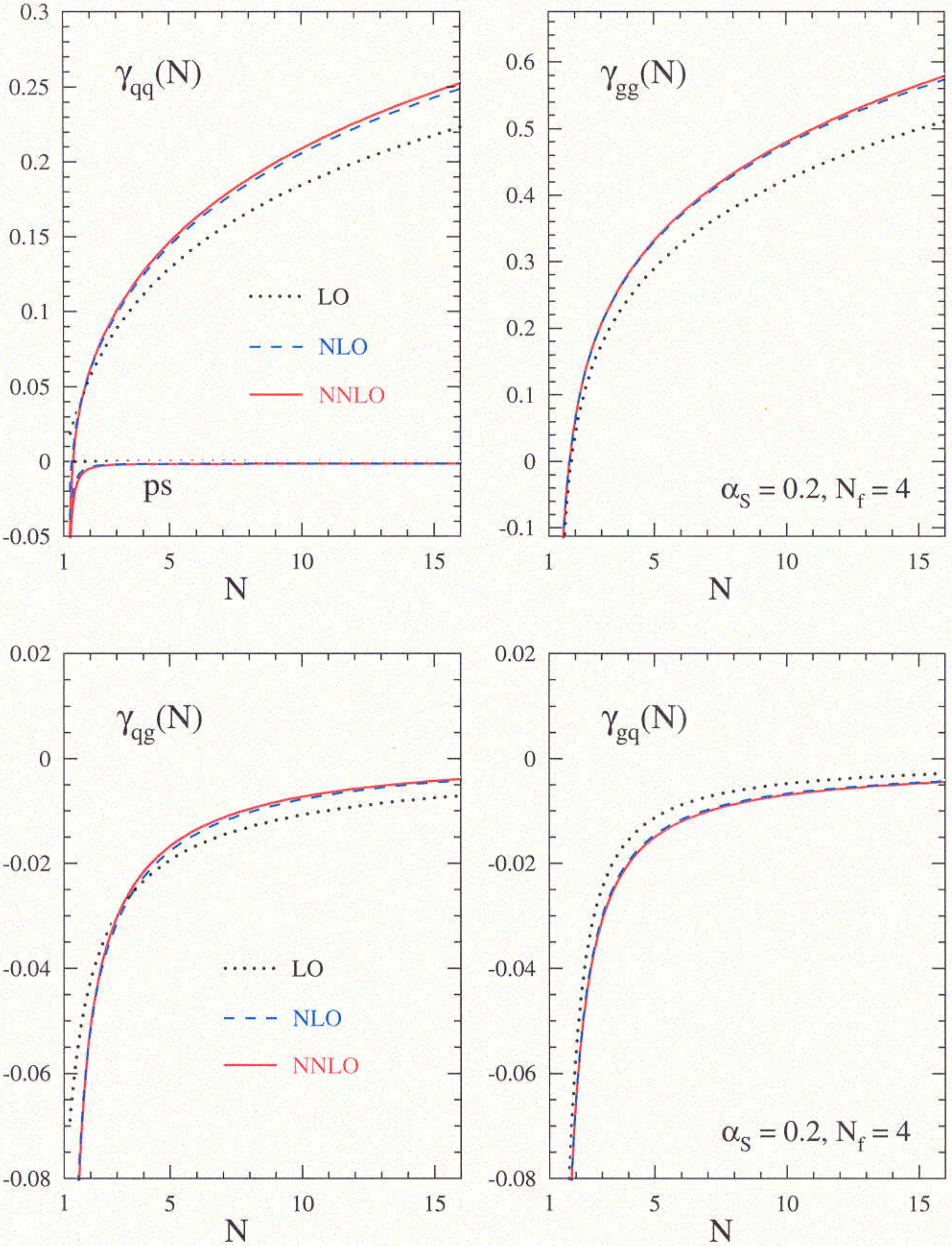
$$+\mathrm{H}_3+6\mathrm{H}_0\zeta_3+8\mathrm{H}_{-3,0}-4x\mathrm{H}_{0,0,0}+(1-x)\Big[\frac{107}{12}\mathrm{H}_1-\frac{5}{6}\mathrm{H}_{1,0}-4\zeta_2+\mathrm{H}_0\zeta_3-8\mathrm{H}_{-2,-1,0}$$

$$-4\mathrm{H}_{-2}\zeta_2+4\mathrm{H}_{-2,0,0}-4\mathrm{H}_1\zeta_2+\frac{7}{2}\mathrm{H}_{1,0,0}-\frac{7}{12}\mathrm{H}_{1,1}+\mathrm{H}_{1,1,0}+\mathrm{H}_{1,1,1}\Big]+(1+x)\Big[\frac{5}{4}\mathrm{H}_2+\frac{33}{8}$$

$$-\frac{99}{4}\mathrm{H}_{0,0}-8\mathrm{H}_{2,0}-\frac{541}{24}\mathrm{H}_0-4\mathrm{H}_{2,1}-\frac{3}{2}\mathrm{H}_{0,0,0}-2x\zeta_3+\frac{9}{2}\zeta_2{}^2+5\mathrm{H}_0\zeta_2-5\mathrm{H}_3-4\mathrm{H}_{-1}\zeta_2$$

$$-8\mathrm{H}_{-1,-1,0}+\frac{67}{3}\mathrm{H}_{-1,0}+4\mathrm{H}_{-1,0,0}+2\mathrm{H}_{0,0}\zeta_2-2\mathrm{H}_{0,0,0,0}-4\mathrm{H}_2\zeta_2+3\mathrm{H}_{2,0,0}+2\mathrm{H}_{2,1,0}$$

$$+2\mathrm{H}_{2,1,1}+\mathrm{H}_{3,1}-2\mathrm{H}_4\Big]+\frac{1}{16}\delta(1-x)\Big)$$



γqq(N)
LO
NLO
NNLO
ps
γgg(N)
αS = 0.2, Nf = 4
γqg(N)
γgq(N)
N

Large-x behaviour

Structure at three loops

$$P^{(2)}_{\mathrm{aa},x\to 1}(x) = \frac{A_3^{\mathrm{a}}}{(1-x)_+} + B_3^{\mathrm{a}}\delta(1-x) + C_3^{\mathrm{a}}\ln(1-x) + \mathcal{O}(1)$$

Korchemsky (89)

Leading terms $\longrightarrow$ soft-gluon resummation

$$\begin{aligned} A_3^q \; = \; & + 16\, C_F C_A^2 \left(\frac{245}{24} - \frac{67}{9}\,\zeta_2 + \frac{11}{6}\,\zeta_3 + \frac{11}{5}\,\zeta_2^2\right) \\ & + 16\, C_F C_A n_f \left(-\frac{209}{108} + \frac{10}{9}\,\zeta_2 - \frac{7}{3}\,\zeta_3\right) \\ & + 16\, C_F^2 n_f \left(-\frac{55}{24} + 2\,\zeta_3\right) + 16\, C_F n_f^2 \left(-\frac{1}{27}\right) \end{aligned}$$

$C_A A_3^{\mathrm{q}} = C_F A_3^{\mathrm{g}}$. n_f part of A_3^{q} also by C. Berger (02)

n_f^2 part Gracey (94)

Subleading logarithms: surprising relation

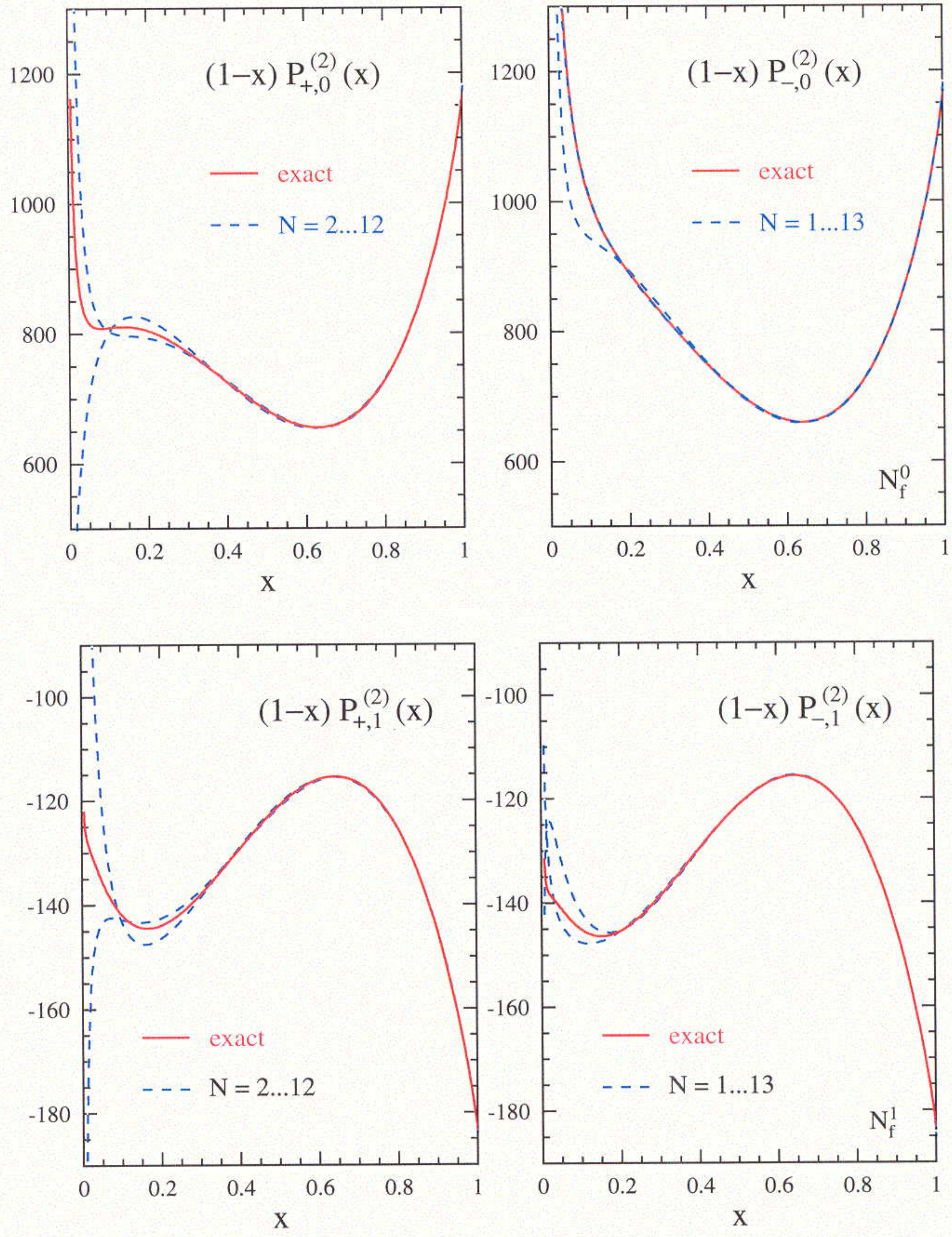
$$\begin{array}{lll} C_1^{\mathrm{q}} = 0\,, & C_2^{\mathrm{q}} = 4\, C_F\, A_1^{\mathrm{q}}\,, & C_3^{\mathrm{q}} = 8\, C_F\, A_2^{\mathrm{q}} \\ C_1^{\mathrm{g}} = 0\,, & C_2^{\mathrm{g}} = 4\, C_A\, A_1^{\mathrm{g}}\,, & C_3^{\mathrm{g}} = 8\, C_A\, A_2^{\mathrm{g}} \end{array}$$

Suggests yet unexplored structure

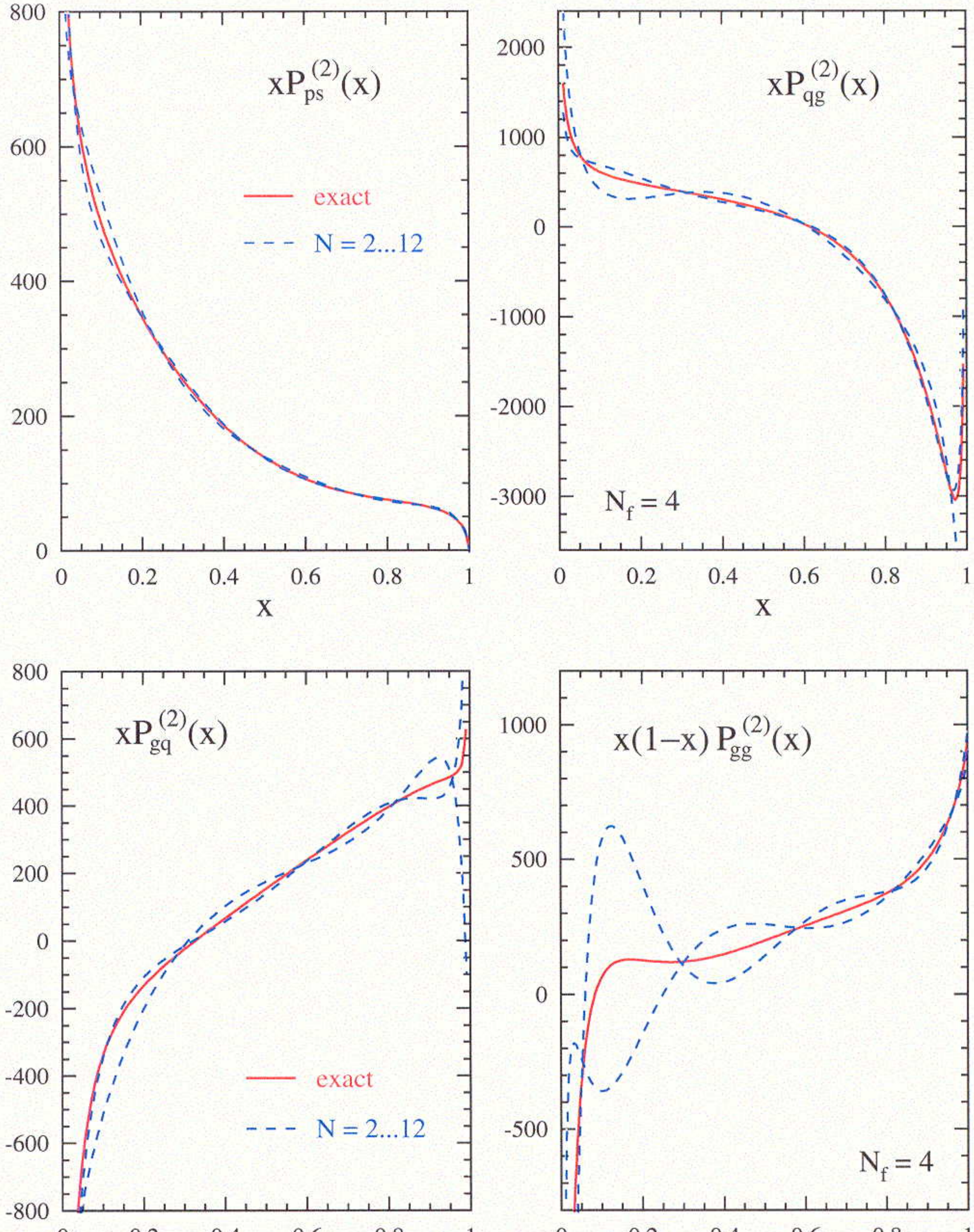
Off-diagonal contributions

$$P^{(2)}_{ab,x\to 1}(x) = \sum_{i=0}^{3} D_i^{\mathrm{ab}} \ln^{4-i}(1-x) + \mathcal{O}(1)$$

$D_{0,1,2}^{\mathrm{ab}} = 0$ for $C_A \equiv N_c = C_F = n_f$: SUSY relation



$(1-x)\,P^{(2)}_{+,0}(x)$
exact
N = 2...12
$(1-x)\,P^{(2)}_{-,0}(x)$
exact
N = 1...13
N_f^0
$(1-x)\,P^{(2)}_{+,1}(x)$
exact
N = 2...12
$(1-x)\,P^{(2)}_{-,1}(x)$
exact
N = 1...13
N_f^1
x
x
x
x



$xP_{ps}^{(2)}(x)$
exact
N = 2...12
$xP_{qg}^{(2)}(x)$
$N_f = 4$
$xP_{gq}^{(2)}(x)$
exact
N = 2...12
$x(1-x)\,P_{gg}^{(2)}(x)$
$N_f = 4$
x
x
x
x

Small-x : non-singlet case

Structure at three loops

$$P^{(2)i}_{x\to 0}(x) \;=\; D_0^i \ln^4 x \;+\; \ldots \;+\; D_3^i \ln x \;+\; \mathcal{O}(1)$$

Generally terms up to $\ln^{2k} x$ at order α_s^{k+1}

Coefficients for 'plus' and 'minus' cases

$$\begin{aligned}
D_0^+ &\cong 1.58025 \\
D_1^+ &\cong 29.6296 - 2.37037\, n_f \\
D_2^+ &\cong 295.042 - 32.1975\, n_f + 0.592592\, n_f^2 \\
D_3^+ &\cong 1261.11 - 152.597\, n_f + 4.345679\, n_f^2
\end{aligned}$$

$$\begin{aligned}
D_0^- &\cong 1.43210 \\
D_1^- &\cong 35.5556 - 3.16049\, n_f \\
D_2^- &\cong 399.205 - 39.7037\, n_f + 0.592592\, n_f^2 \\
D_3^- &\cong 1465.93 - 172.693\, n_f + 4.345679\, n_f^2
\end{aligned}$$

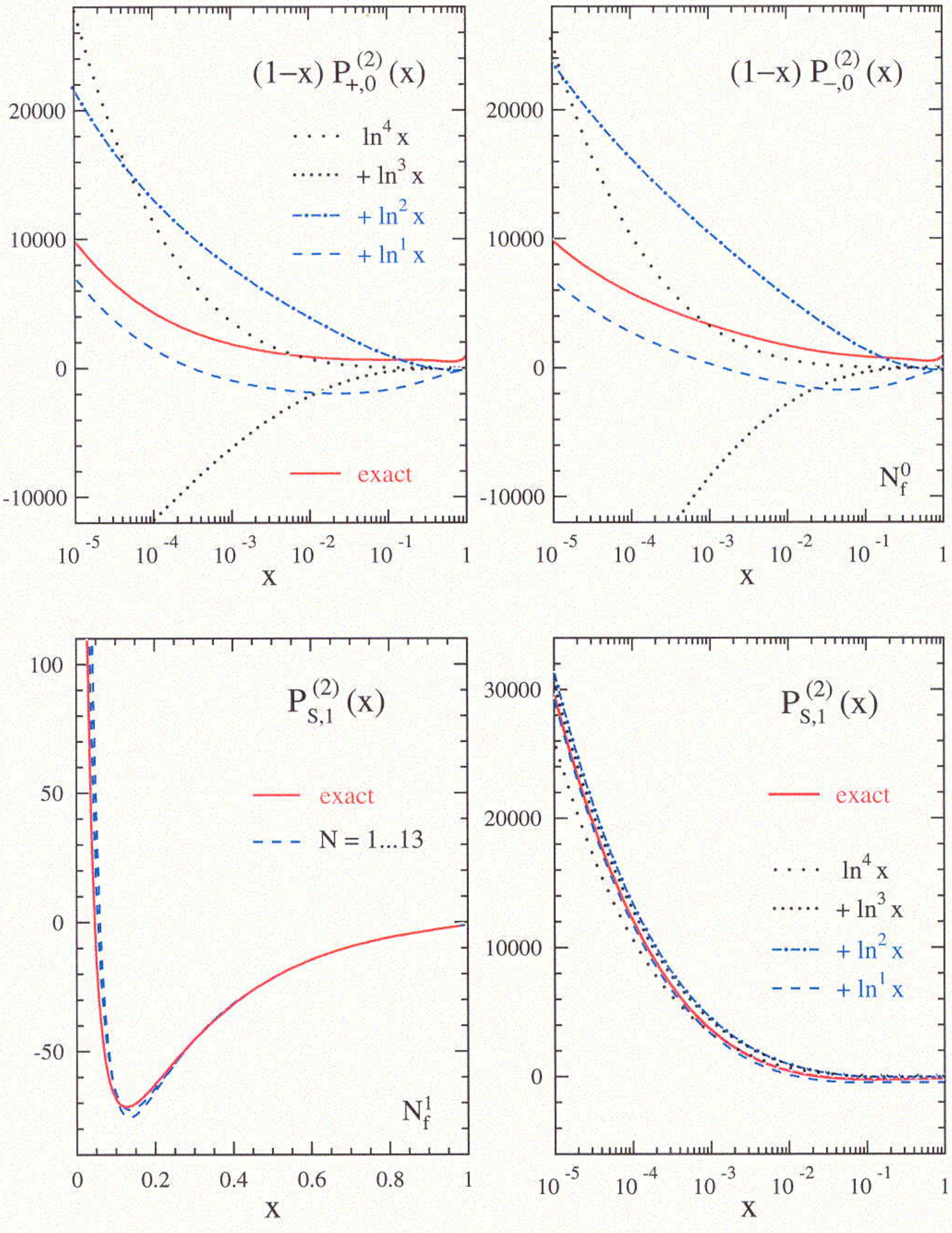
$D_0^{\pm}$: Kirschner, Lipatov (83), Blümlein, A.V. (95)

Large logarithms often have small coefficients

New $d_{abc}d_{abc}$ contribution to 'valence' case

$$\begin{aligned}
D_0^{S} &\cong +1.48148\, n_f \\
D_1^{S} &\cong -2.96296\, n_f \\
D_2^{S} &\cong +6.89182\, n_f \\
D_3^{S} &\cong +178.030\, n_f
\end{aligned}$$

Presence of $\ln^4 x$ term unpredicted



$(1-x)\,P^{(2)}_{+,0}(x)$
$\ln^4 x$
$+\ln^3 x$
$+\ln^2 x$
$+\ln^1 x$
exact
20000
10000
0
-10000
10^{-5} 10^{-4} 10^{-3} 10^{-2} 10^{-1} 1
x
$(1-x)\,P^{(2)}_{-,0}(x)$
N_f^0
$P^{(2)}_{S,1}(x)$
exact
N = 1...13
N_f^1
100
50
0
-50
0 0.2 0.4 0.6 0.8 1
x
$P^{(2)}_{S,1}(x)$
exact
$\ln^4 x$
$+\ln^3 x$
$+\ln^2 x$
$+\ln^1 x$
30000
20000
10000
0

Small-x: flavour-singlet case

Structure at three loops

$$P^{(2)}_{\mathrm{ab},x\to 0}(x) \; = \; E_1^{\mathrm{ab}} \, \frac{\ln x}{x} \; + \; E_2^{\mathrm{ab}} \, \frac{1}{x} \; + \; \mathcal{O}(\ln^4 x)$$

Generally terms up to $x^{-1} \ln^k x$ at order α_s^{k+1}

Coefficients of $1/x$ contributions

$$E_1^{\mathrm{qq}} \;\cong\; -132.741\, n_f$$
$$E_2^{\mathrm{qq}} \;\cong\; -505.999\, n_f + 3.16049\, n_f^2$$

$$E_1^{\mathrm{qg}} \;\cong\; -298.667\, n_f$$
$$E_2^{\mathrm{qg}} \;\cong\; -1268.28\, n_f + 4.57613\, n_f^2$$

$$E_1^{\mathrm{gq}} \;\cong\; +1189.27 + 71.0825\, n_f$$
$$E_2^{\mathrm{gq}} \;\cong\; +6163.11 - 46.4075\, n_f - 2.37037\, n_f^2$$

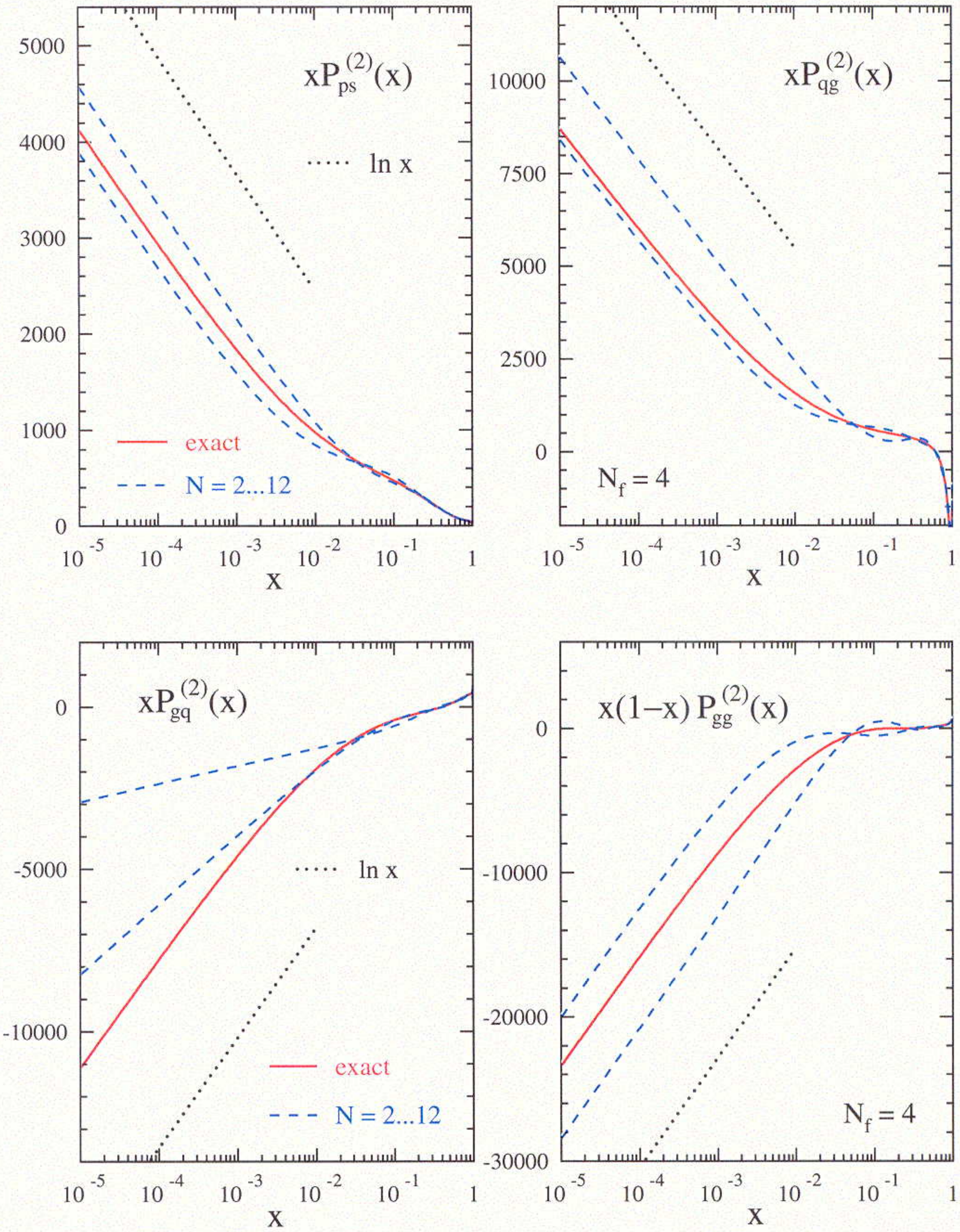
$$E_1^{\mathrm{gg}} \;\cong\; +2675.85 + 157.269\, n_f$$
$$E_2^{\mathrm{gg}} \;\cong\; +14214.2 + 182.958\, n_f - 2.79835\, n_f^2$$

E_1^{qa} Catani, Hautmann (94), E_1^{gg} Fadin, Lipatov (98)

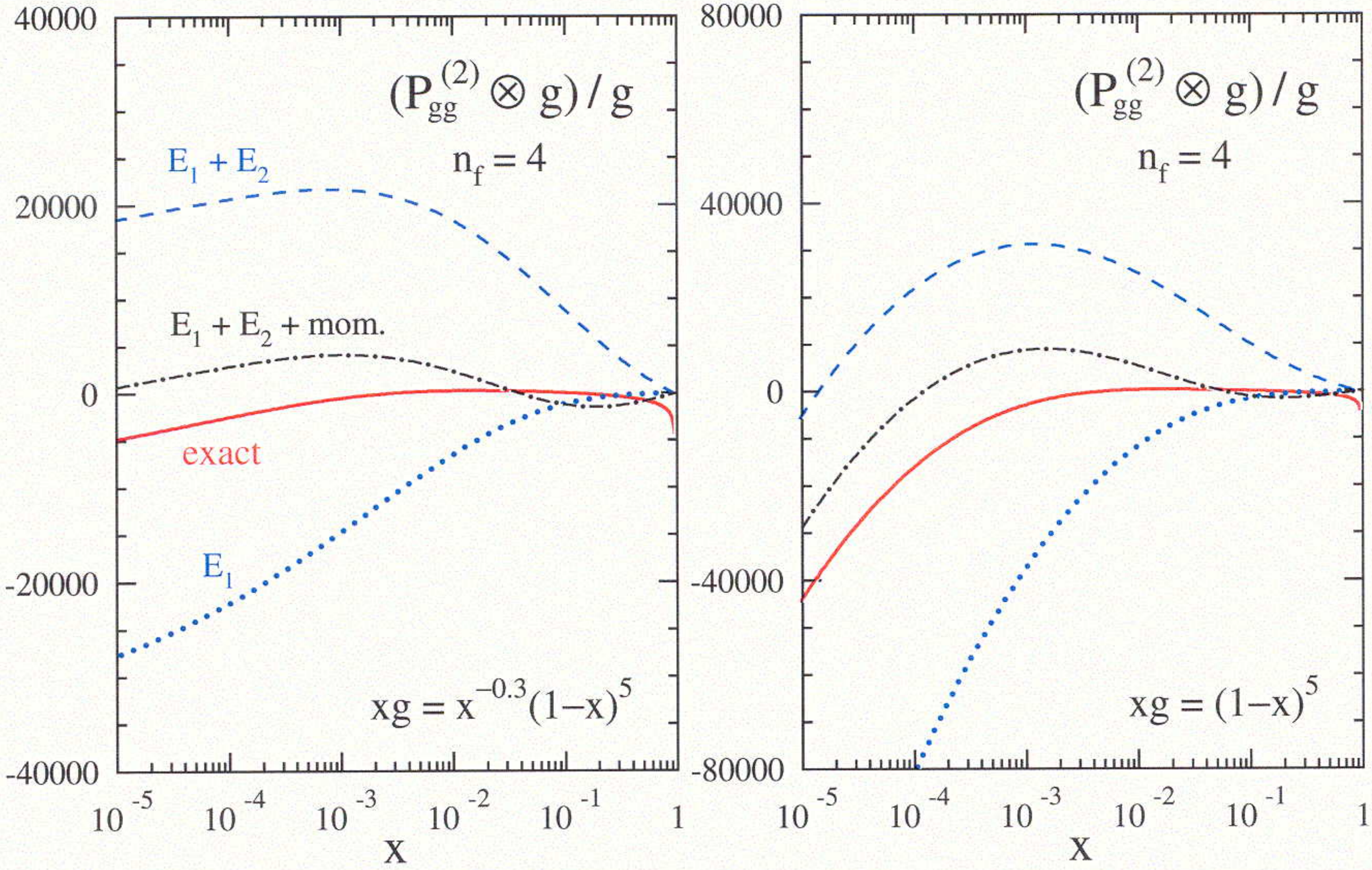
$E_2^{\mathrm{ab}}/E_1^{\mathrm{ab}}$ for $n_f = 3 \ldots 5$: $3.7 \ldots 4.7$ Cf. $\ln 10^4 \simeq 9.2$

Relations between coefficients

$$E_1^{\mathrm{qg}} \;=\; \frac{C_A}{C_F}\, E_1^{\mathrm{qq}}$$
$$E_1^{\mathrm{gg}} \;=\; \frac{C_A}{C_F}\, E_1^{\mathrm{gq}} - \frac{8}{3}\, n_f$$



$xP_{ps}^{(2)}(x)$
ln x
exact
N = 2...12
$xP_{qg}^{(2)}(x)$
$N_f = 4$
$xP_{gq}^{(2)}(x)$
ln x
exact
N = 2...12
$x(1-x)\,P_{gg}^{(2)}(x)$
$N_f = 4$
x



$(P_{gg}^{(2)} \otimes g)/g$
$n_f = 4$
$E_1 + E_2$
$E_1 + E_2$ + mom.
exact
E_1
$xg = x^{-0.3}(1-x)^5$
$xg = (1-x)^5$
x
40000
20000
0
-20000
-40000
80000
40000
-40000
-80000
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
1

Parton evolution at NNLO

Illustrated by logarithmic derivatives $\dot{f} = d \ln f / d \ln \mu_f^2$

Order-independent model distributions

$$\begin{aligned} xq_{\rm ns}(x,\mu_0^2) &= x^{0.5}(1-x)^3 \\ xq_{\rm s}(x,\mu_0^2) &= 0.6\, x^{-0.3}(1-x)^{3.5}(1+5.0\, x^{0.8}) \\ xg(x,\mu_0^2) &= 1.6\, x^{-0.3}(1-x)^{4.5}(1-0.6\, x^{0.3}) \end{aligned}$$

for

$$\alpha_{\rm S}(\mu_0^2) = 0.2\,, \quad n_f = 4$$

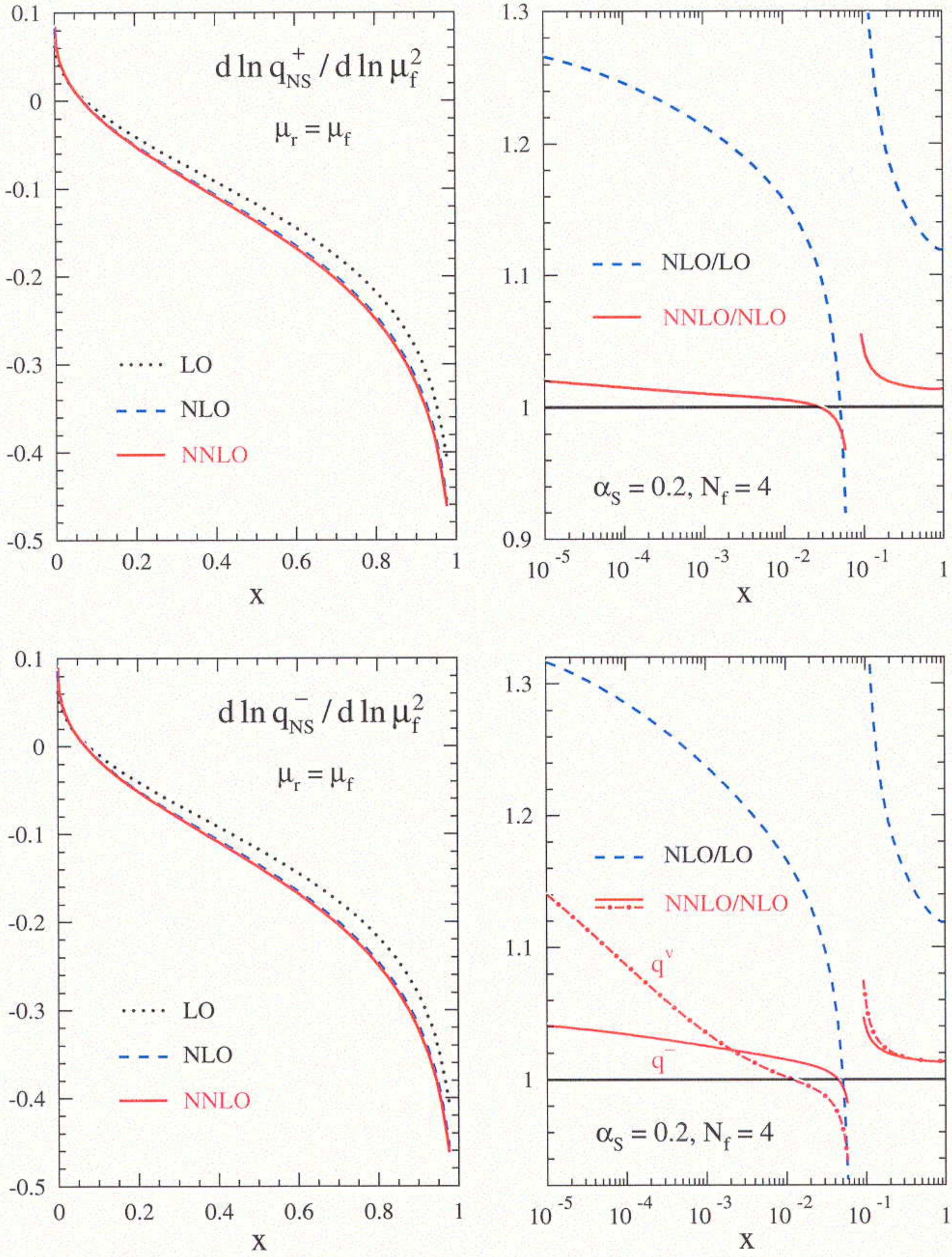
Renormalization scale dependence

$$\begin{aligned} P_{\rm ab}^{\rm NNLO}(\mu_f,\mu_r) \; = & \\ & a_{\rm s}(\mu_r^2)\; P_{\rm ab}^{(0)} \\ & + a_{\rm s}^2(\mu_r^2)\left(P_{\rm ab}^{(1)} - \beta_0\, P_{\rm ab}^{(0)} L\right) \\ & + a_{\rm s}^3(\mu_r^2)\left(P_{\rm ab}^{(2)} - \left\{\beta_1 P_{\rm ab}^{(0)} + 2\beta_0\, P_{\rm ab}^{(1)}\right\} L + \beta_0^2\, P_{\rm ab}^{(0)} L^2\right) \end{aligned}$$

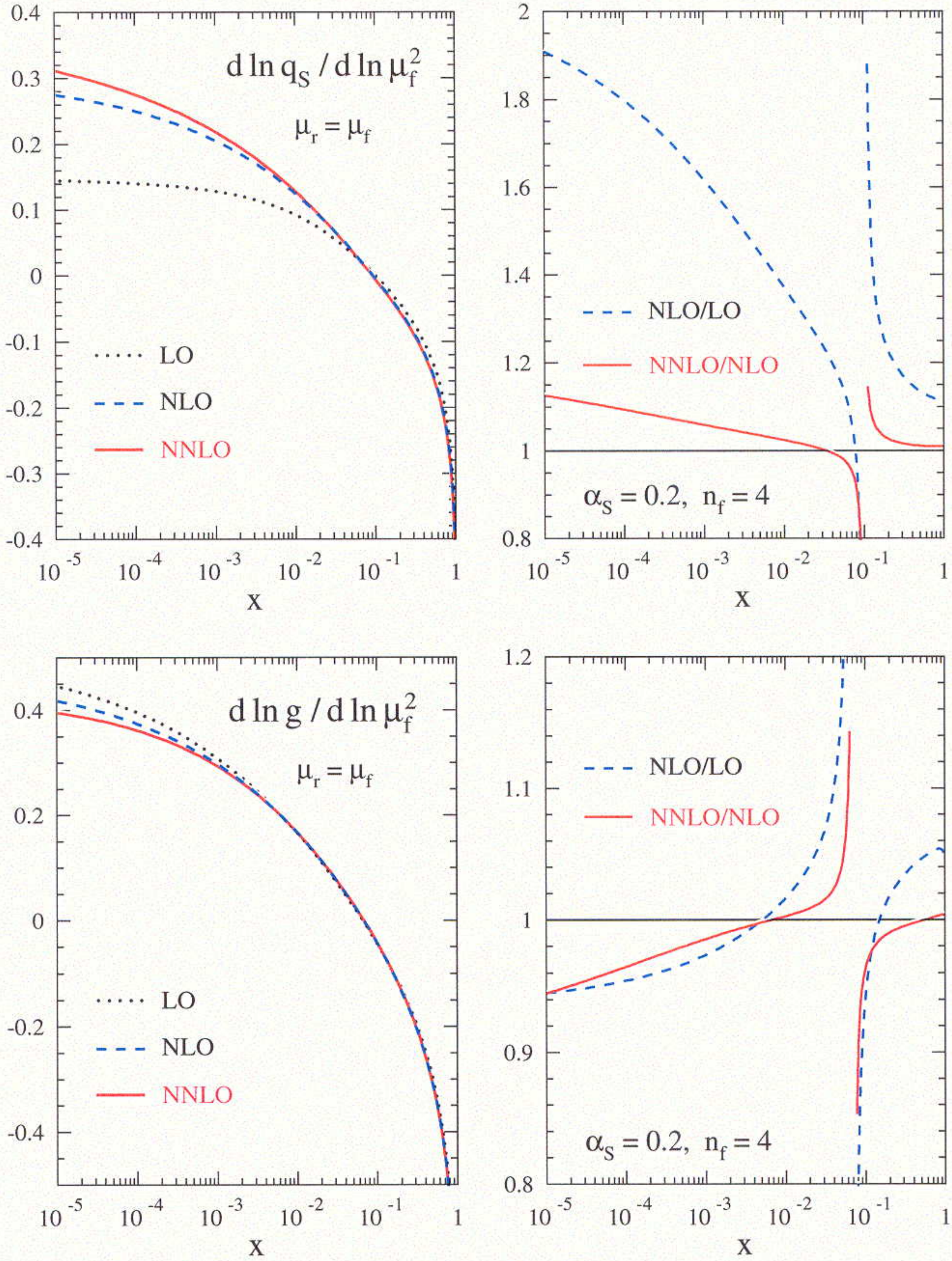
Abbreviations: $a_{\rm S} = \alpha_{\rm S}/(4\pi)$, $L = \ln(\mu_f^2/\mu_r^2)$

Conventional estimate for relative scale uncertainty

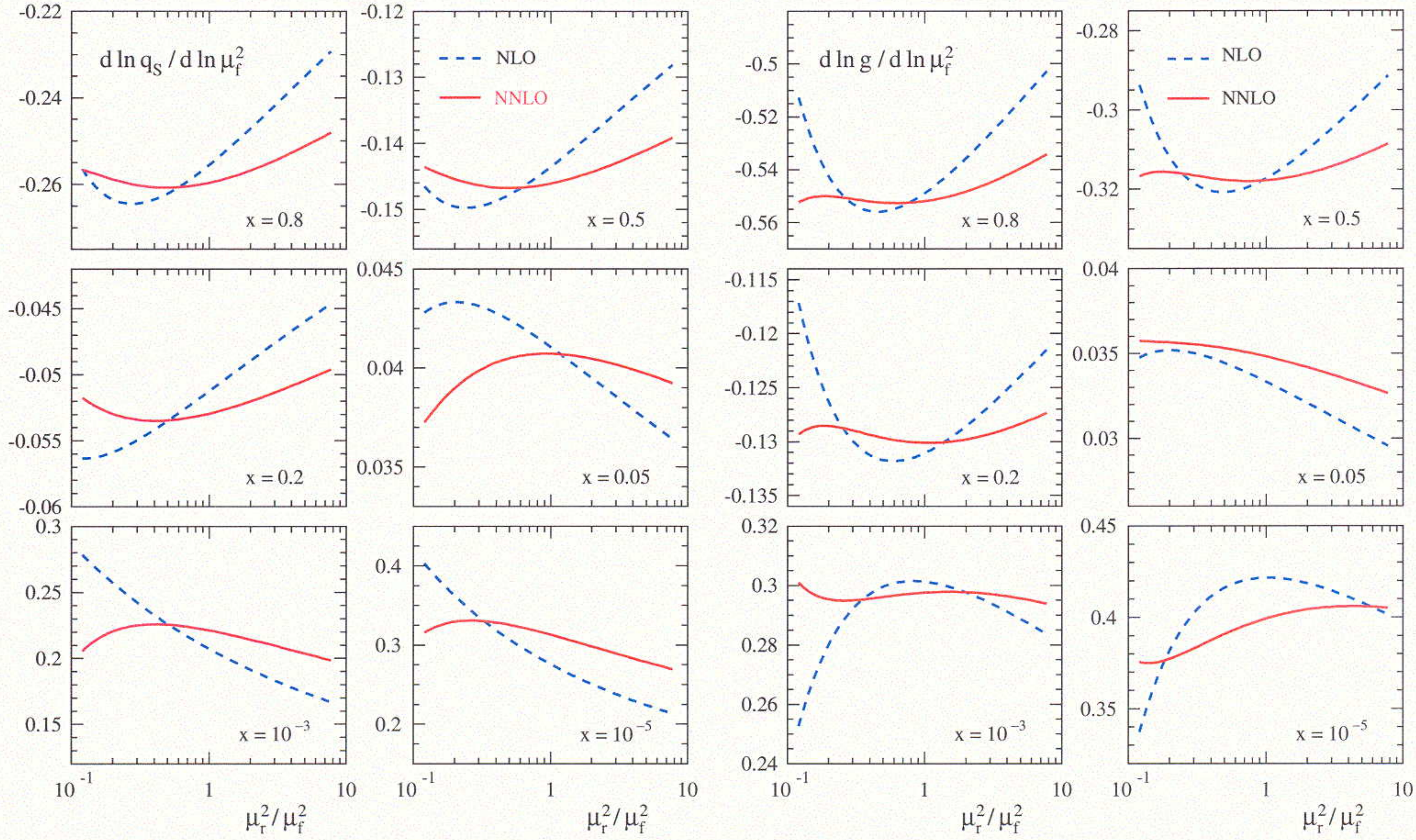
$$\Delta \dot{f} \;=\; \frac{\max\, \dot{f} - \min\, \dot{f}}{2\,|\,\text{average}\, \dot{f}|}\;,\;\; \mu_r^2 = \frac{1}{4}\,\mu_f^2 \ldots 4\,\mu_f^2$$



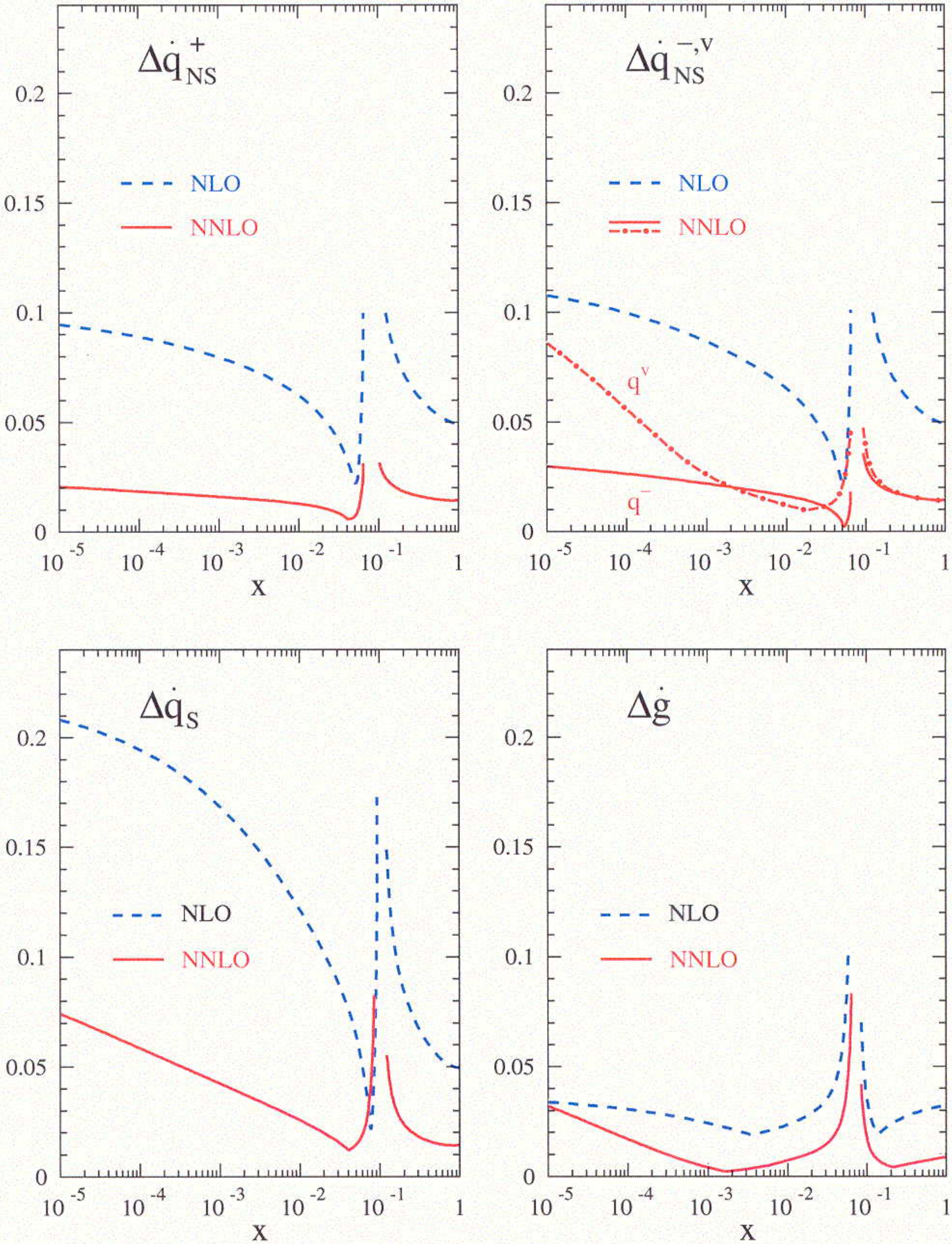
$d \ln q_{NS}^{+} / d \ln \mu_f^2$
$\mu_r = \mu_f$
LO
NLO
NNLO
x
NLO/LO
NNLO/NLO
$\alpha_S = 0.2, N_f = 4$
x
$d \ln q_{NS}^{-} / d \ln \mu_f^2$
$\mu_r = \mu_f$
LO
NLO
NNLO
x
NLO/LO
NNLO/NLO
q^v
q^-
$\alpha_S = 0.2, N_f = 4$
x



d ln q_S / d ln μ_f^2
μ_r = μ_f
LO
NLO
NNLO
NLO/LO
NNLO/NLO
α_S = 0.2, n_f = 4
d ln g / d ln μ_f^2
x



$d \ln q_S / d \ln \mu_f^2$
$d \ln g / d \ln \mu_f^2$
NLO
NNLO
x = 0.8
x = 0.5
x = 0.2
x = 0.05
$x = 10^{-3}$
$x = 10^{-5}$
μ_r^2 / μ_f^2



$\Delta\dot{q}^{+}_{NS}$
NLO
NNLO
$\Delta\dot{q}^{-,v}_{NS}$
NLO
NNLO
q^{v}
q^{-}
$\Delta\dot{q}_{S}$
NLO
NNLO
$\Delta\dot{g}$
NLO
NNLO
0
0.05
0.1
0.15
0.2
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
1
x

Summary

Unpolarized three-loop (NNLO) splitting functions calculated via all-N expressions in moment space

- Check of the Form manipulations at all stages
  Fall back to small fixed N $\rightarrow$ Mincer program
  Gorishny et al. (1989); Larin et al. (91)
- Barely possible with present resources
  ~ 10 person-years, several CPU years, Form 3.1
  10^4 diagrams, $\sim 10^5$ tabulated integrals ($>$ 3 GB)
- To come: three-loop coefficient functions
  Planned : polarized case, photon structure

Agreement with all known partial results

- Fixed low-integer moments
  Larin et al. (94,97); Retey, Vermaseren (00)
- Large-n_f limits
  Gracey (94); Bennett, Gracey (97)
- Large-x structure and limits
  Korchemsky (89), C. Berger (02)
- Small-x limits — Catani, Hautmann (94); Blümlein, A.V. (95); Fadin, Lipatov (98)
- Approximations — van Neerven, A.V. (00)

Summary

NNLO effects on PDF evolution small at $x \gtrsim 10^{-3}$

Large x for $\alpha_{\rm S} = 0.2$: corrections, μ_r variation: $< 2\%$, 3-loop effects ~ 8 times smaller than 2-loop corr.

Evolution perturbative down to rather large $\alpha_{\rm S}$

Small-x: corrections increase with decreasing x

Three lessons at small momentum fractions

- New colour factors can give dominant higher-order terms not predicted by resummations
- The leading terms for $x \to 0$ do not dominate the splitting functions at exp. relevant x-values
- Convolutions $P \otimes f$ entering the evolution are sensitive even to terms power-suppressed in x

Small-x constraints need to be complemented by large-x information: at least several moments

Relations found between diagonal large-x terms

Coeff. of $\ln(1-x) \sim$ coeff. of $1/(1-x)$ at order $n-1$