Preserving and changing the type of convergence of a series

Peter Eliaš

Mathematical Institute, Slovak Academy of Sciences, Košice, Slovakia

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Functions preserving the convergence of series

Theorem (R. Rado)

Let $f : \mathbb{R} \to \mathbb{R}$. The following conditions are equivalent.

1. f *preserves the convergence of series, i.e., for every* $\{x_n\}_{n\in\mathbb{N}}$, *if* $\sum x_n$ *converges then* $\sum f(x_n)$ *converges,*
2. $\exists a \in \mathbb{R}\ \exists \delta > 0\ \forall x \in (-\delta, \delta)\ f(x) = ax.$

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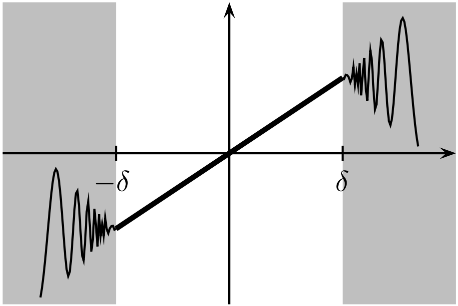
A graph of convergence preserving function



Proof of Rado's theorem

Lemma

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at 0. Then the following conditions are equivalent.

1. $f(x + y) = f(x) + f(y)$ *holds on some nbhd of* 0,
2. *there is* $a \in \mathbb{R}$ *such that* $f(x) = ax$ *holds on some nbhd of* 0.

Proof. $2 \Rightarrow 1$ is trivial. To prove $1 \Rightarrow 2$:

- show that f is continuous on some nbhd of 0,
- prove that on some nbhd of 0, $f(rx) = rf(x)$ holds for every rational r,
- use the continuity of f to show that $f(rx) = rf(x)$ holds true on some nbhd of 0 for all $r \in \mathbb{R}$.

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Some references

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Results of J. Borsík

Let A, B be families of sequences of real numbers.
Denote $F(A, B)$ the family of all functions $f : \mathbb{R} \to \mathbb{R}$ mapping every sequence $\{x_n\}_{n\in\mathbb{N}} \in A$ to a sequence $\{f(x_n)\}_{n\in\mathbb{N}} \in B$.

Consider the following families:

$$
\begin{aligned}
C &= \left\{\{x_n\}_{n\in\mathbb{N}} : \sum x_n \text{ converges}\right\},\\
AC &= \left\{\{x_n\}_{n\in\mathbb{N}} : \sum x_n \text{ absolutely converges}\right\},\\
RC &= \left\{\{x_n\}_{n\in\mathbb{N}} : \sum x_n \text{ relatively converges}\right\} = C \setminus AC,\\
D &= \left\{\{x_n\}_{n\in\mathbb{N}} : \sum x_n \text{ diverges}\right\}.
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In Borsík's paper, all families $F(A, B)$ are characterized, for $A \in \{C, AC, RC, D\}$, except $F(D, D)$ and $F(RC, D)$.

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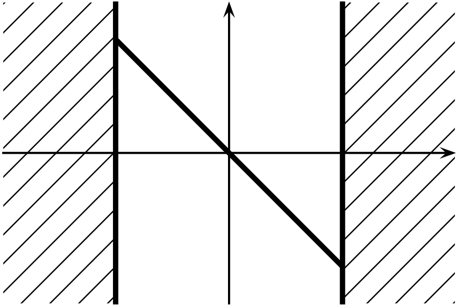
Theorem (Rado, Borsík)

1. $F(C,C) = F(RC,C) = \{f : \exists a\ \exists b > 0\ f \subseteq N(a,b)\}$,
2. $F(RC,RC) = \{f : \exists a \neq 0\ \exists b > 0\ f \subseteq N(a,b)\}$,
3. $F(C,AC) = F(RC,AC) = \{f : \exists b > 0\ f \subseteq N(0,b)\}$,

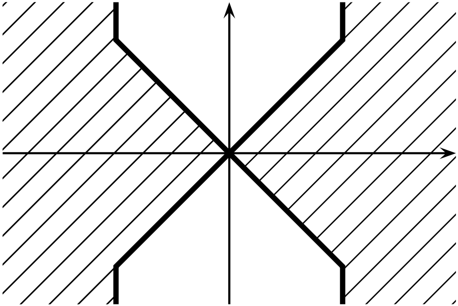
where $N(a,b) = \{(x,y) \in \mathbb{R}^2\ :\ y = ax\ \vee\ |x| \geq b\}$.



Theorem (Borsík, Červeňanský, Šalát)

4. $F(AC,C) = F(AC,AC) = \{f : \exists a \geq 0\ \exists b > 0\ f \subseteq X(a,b)\}$,

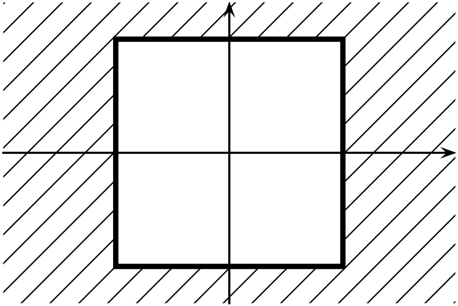
where $X(a,b) = \{(x,y) \in \mathbb{R}^2 : |y| \leq a\,|x| \ \vee\ |x| \geq b\}$.



Theorem (Borsík)

5. $F(C,D) = F(AC,D) = \{f : \exists b > 0\ f \subseteq O(b)\}$,

where $O(b) = \{(x,y) \in \mathbb{R}^2 : |x| \geq b \ \vee \ |y| \geq b\}$.



Fact

6. $F(D,C) = F(D,AC) = \{f : \forall x \ f(x) = 0\}$,
7. $F(C,RC) = F(AC,RC) = F(D,RC) = \emptyset$.

In all cases, $F(A,B)$ was characterized by the condition of the form

$$f \in F(A,B) \Longleftrightarrow \exists X \in \mathcal{C} \ f \subseteq X,$$

where $\mathcal{C}$ is some family of closed subsets of $\mathbb{R}^2$.

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Characterization by a family of closed sets

Fact

Let $A \in \{C, AC, RC, D\}$.
If $\{x_n\}_{n\in\mathbb{N}} \in A$ *and* $\{y_n\}_{n\in\mathbb{N}} \in AC$ *then* $\{x_n + y_n\}_{n\in\mathbb{N}} \in A$.

Corollary

Let $A, B \in \{C, AC, RC, D\}$.

1. *If* $f \in F(A,B)$ *and* $g \subseteq \mathrm{cl}(f)$ *then* $g \in F(A,B)$.
2. *There exists* $\mathcal{C}$, *a family of closed subsets of* $\mathbb{R}^2$, *such that* $f \in F(A,B) \iff \exists X \in \mathcal{C}\ f \subseteq X$.

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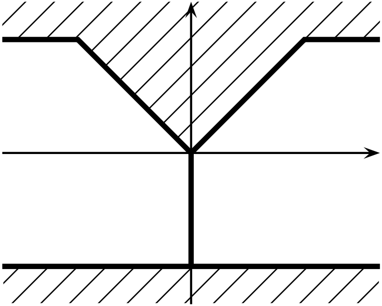
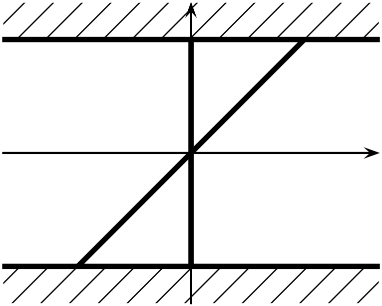
Functions preserving the divergence of series

Theorem (P.E., Stará Lesná 2008)

$$F(D,D) = \{f : \exists a\ \exists b > 0\ (f \subseteq Y(a,b)\ \vee\ f \subseteq Z(a,b))\},$$

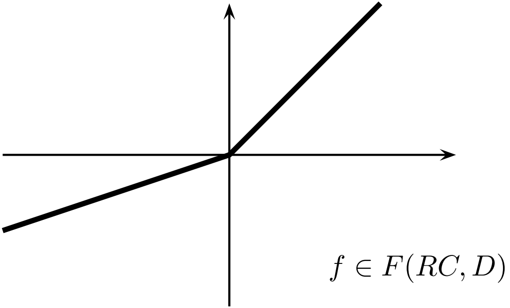
where $Y(a,b) = \{(x,y) \in \mathbb{R}^2 : x = 0\ \vee\ |y| \geq b\ \vee\ |x| \leq ay\}$ and $Z(a,b) = \{(x,y) \in \mathbb{R}^2 : x = 0\ \vee\ |y| \geq b\ \vee\ x = ay\}$.





Example

Let $f(x) = ax$ if $x \geq 0$ and $f(x) = bx$ if $x < 0$. Then $f \in F(RC, D) \iff a \neq b$.



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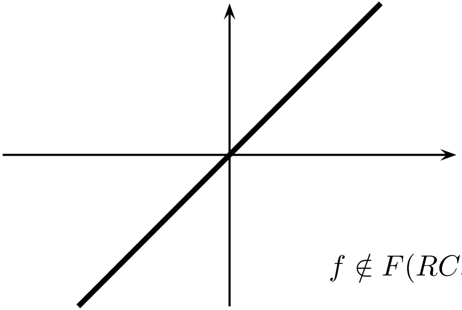
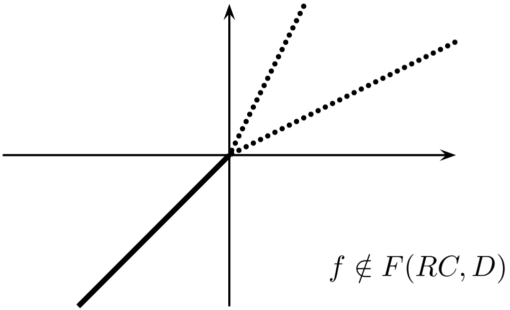


$f \notin F(RC, D)$

Example

Let $f(x) = \begin{cases} x/2 \text{ if } x \geq 0 \text{ and } x \in \mathbb{Q}, \\ 2x \text{ if } x \geq 0 \text{ and } x \notin \mathbb{Q}, \\ x \text{ if } x < 0. \end{cases}$

Then $f \notin F(RC, D)$.



Towards the characterization of $F(RC, D)$

Lemma

Let $f : \mathbb{R} \to \mathbb{R}$. The following conditions are equivalent.

1. $f \notin F(RC, D)$,
2. *there exists a sequence $\{S_n\}_{n\in\mathbb{N}}$ of nonempty finite subsets of $\mathbb{R}$ such that*
   - $\lim\limits_{n\to\infty} \sum\limits_{x\in S_n} x = \lim\limits_{n\to\infty} \sum\limits_{x\in S_n} f(x) = 0,$
   - $\lim\limits_{n\to\infty} \max\limits_{x\in S_n} |x| = \lim\limits_{n\to\infty} \max\limits_{x\in S_n} |f(x)| = 0,$
   - $\liminf\limits_{n\to\infty} \sum\limits_{x\in S_n} |x| > 0.$

For $f : \mathbb{R} \to \mathbb{R}$, $\varepsilon > 0$ denote

$$D_f^+(\varepsilon) = (0, \varepsilon) \cap f^{-1}[(0, \varepsilon)], \quad D_f^-(\varepsilon) = (-\varepsilon, 0) \cap f^{-1}[(-\varepsilon, 0)],$$

$$H_f^+(\varepsilon) = \operatorname{cl}\left\{ \sum_{x \in S} f(x) / \sum_{x \in S} x : S \text{ is a nonempty finite subset of } D_f^+ \right\},$$

$$H_f^-(\varepsilon) = \operatorname{cl}\left\{ \sum_{x \in S} f(x) / \sum_{x \in S} x : S \text{ is a nonempty finite subset of } D_f^- \right\}.$$

Fact

$H_f^+(\varepsilon)$ and $H_f^i(\varepsilon)$ are closed connected sets, i.e., intervals.

Let $H_f^+ = \bigcap_{\varepsilon > 0} H_f^+(\varepsilon)$, $H_f^- = \bigcap_{\varepsilon > 0} H_f^-(\varepsilon)$.

For $f:\mathbb{R}\to\mathbb{R}$, $\varepsilon>0$ denote

$$D_f^+(\varepsilon)=(0,\varepsilon)\cap f^{-1}[(0,\varepsilon)],\quad D_f^-(\varepsilon)=(-\varepsilon,0)\cap f^{-1}[(-\varepsilon,0)],$$

$$H_f^+(\varepsilon)=\mathrm{cl}\left\{\sum_{x\in S}f(x)/\sum_{x\in S}x : S \text{ is a nonempty finite subset of } D_f^+\right\},$$

$$H_f^-(\varepsilon)=\mathrm{cl}\left\{\sum_{x\in S}f(x)/\sum_{x\in S}x : S \text{ is a nonempty finite subset of } D_f^-\right\}.$$

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$H_f^+(\varepsilon)$ and $H_f^i(\varepsilon)$ are closed connected sets, i.e., intervals.

Let $H_f^+=\bigcap_{\varepsilon>0}H_f^+(\varepsilon)$, $H_f^-=\bigcap_{\varepsilon>0}H_f^-(\varepsilon)$.

Theorem

Let $f : \mathbb{R} \to \mathbb{R}$. The following conditions are equivalent.

1. $f \notin F(RC, D)$,
2. $H_f^+ \cap H_f^- \neq \emptyset$,
3. *for every $\varepsilon > 0$ there are nonempty finite sets $S_1, S_2 \subseteq D_f^+$, $R \subseteq D_f^-$ such that*

$$\sum_{x \in S_1} f(x) / \sum_{x \in S_1} x \leq \sum_{x \in R} f(x) / \sum_{x \in R} x \leq \sum_{x \in S_2} f(x) / \sum_{x \in S_2} x.$$