A Galois connection related to the uniform approximability of continuous functions

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Motivation

$\mathbb{T}$ – the unit circle

- $\mathbb{T} = \mathbb{R}/\sim$, where $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$
- addition modulo integers, quotient topology
- invariant metric $d(x, y) = \|x - y\|$, where $\|x\| = \min\{|x - k| : k \in \mathbb{Z}\}$

$C(X, Y)$ – space of all continuous functions $f : X \to Y$ with the topology of uniform convergence

$C(X)$ – family of all closed subsets of X

characters of $\mathbb{T}$ – continuous group homomorphisms $\chi \in C(\mathbb{T}, \mathbb{T})$
these are exactly the functions $\chi_n(x) = nx$ for $n \in \mathbb{Z}$

Definition (Hewitt, Kakutani (1960), Rudin (1962))

A set $E \in C(\mathbb{T})$ is a Kronecker set if

$$(\forall f \in C(\mathbb{T},\mathbb{T}))(\forall \varepsilon > 0)(\exists n)(\forall x \in E)\ \|\chi_n(x) - f(x)\| < \varepsilon.$$

Definition (Kahane (1969))

A set $E \in C(\mathbb{T})$ is a Dirichlet set if

$$(\forall \varepsilon > 0)(\exists n \neq 0)(\forall x \in E)\ \|\chi_n(x)\| < \varepsilon.$$

- we can assume that n is arbitrarily large (in both definitions)
- on a Kronecker set, each continuous function is a uniform limit of a sequence of characters
- on a Dirichlet set, zero function is a uniform limit of a sequence of characters

$\mathcal{K}$, $\mathcal{D}$ – families of all Kronecker and Dirichlet sets

For $\mathcal{F} \subseteq C(\mathbb{T},\mathbb{T})$ and $\mathcal{E} \subseteq C(\mathbb{T})$ define

$$F(\mathcal{E}) = \big\{f \in C(\mathbb{T},\mathbb{T}) : (\forall E \in \mathcal{E})(\exists \text{ increasing } \{n_k\}_{k\in\mathbb{N}})\ \chi_{n_k} \rightrightarrows f \text{ on } E\big\},$$
$$G(\mathcal{F}) = \big\{E \in C(\mathbb{T}) : (\forall f \in \mathcal{F})(\exists \text{ increasing } \{n_k\}_{k\in\mathbb{N}})\ \chi_{n_k} \rightrightarrows f \text{ on } E\big\}.$$

We obtain

- $\mathcal{K} = G(C(\mathbb{T},\mathbb{T}))$, $F(\mathcal{K}) = C(\mathbb{T},\mathbb{T})$,
- $\mathcal{D} = G(\{\chi_0\}) = G(\{\chi_n : n \in \mathbb{Z}\})$, $F(\mathcal{D}) = \{\chi_n : n \in \mathbb{Z}\}$.

Question

For what pairs $(\mathcal{F},\mathcal{E})$ one can have $\mathcal{F} = F(\mathcal{E})$ and $\mathcal{E} = G(\mathcal{F})$?

Such pairs can be viewed as generalizations of the families of Kronecker and Dirichlet sets.

Theorem

Let $\mathcal{C}_k = \{\chi_n + c : n \in \mathbb{Z},\ c \in \mathbb{T},\ kc = 0\}$, $\mathcal{D}_k = G(\mathcal{C}_k)$, *for* $k \in \mathbb{Z}$. *Then* $F(\mathcal{D}_k) = \mathcal{C}_k$.

sets in $\mathcal{D}_0$ – strongly Dirichlet sets
sets in $\mathcal{D}_k$ (for $k \neq 0$) – k-Dirichlet sets

Lemma

Let $f \in C(\mathbb{T}, \mathbb{T})$.

1. $f \in \{\chi_n : n \in \mathbb{Z}\} \Leftrightarrow (\forall x, y \in \mathbb{T})\ f(x+y) = f(x) + f(y)$
2. $f \in \mathcal{C}_0 \Leftrightarrow (\forall x, y \in \mathbb{T})\ f(2x-y) = 2f(x) - f(y)$
3. $f \in \mathcal{C}_k \Leftrightarrow (\forall x, y \in \mathbb{T})\ f(kx+y) = kf(x) + f(y)$

Question

Are there other pairs $(\mathcal{F}, \mathcal{E})$ satisfying $\mathcal{F} = F(\mathcal{E})$ and $\mathcal{E} = G(\mathcal{F})$?

Yes, but we have no characterization of such pairs.

Definitions

X – topological space, (Y, d) – metric space
$f \in C(X, Y)$, $E \in C(X)$, $\Phi \subseteq C(X, Y)$

Definition

f is uniformly approximable by functions from Φ on a set E if there exists a one-to-one sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ in Φ such that $f_n \rightrightarrows f$ on E.

Fact

1. If f_n, f are continuous and $f_n \rightrightarrows f$ on E (not necessarily closed) then $f_n \rightrightarrows f$ on $\mathrm{cl}(E)$.
2. If X is normal, $E \in C(X)$, and $f \in C(E, \mathbb{R})$, then by Tietze-Urysohn Theorem there exists $g \in C(X, \mathbb{R})$ such that $g \restriction E = f$.
   Holds also for $\mathbb{T}$ in place of $\mathbb{R}$.

Galois connection – a pair of mappings between two partial orders

1. $F : P \to Q, \quad G : Q \to P$
2. $q \leq_Q F(p) \Leftrightarrow p \leq_P G(q)$ for all $p \in P$, $q \in Q$

- compound mappings $\quad FG : P \to P, \quad GF : Q \to Q \quad$ are closure operators (i.e., $p \leq_P FG(p)$, $FG(FG(p)) = FG(p)$ for all $p \in P$)
- $p \in P$ is closed iff $p = G(q)$ for some $q \in Q$
- pairs (p, q) such that $q = F(p)$, $p = G(q)$ form a complete lattice when ordered by $(p, q) \leq (p', q') \Leftrightarrow p \leq_P p' \Leftrightarrow q' \leq_Q q$
- a Galois connection between $(\mathcal{P}(X), \subseteq)$ and $(\mathcal{P}(Y), \subseteq)$ is naturally obtained from a binary relation $R \subseteq X \times Y$:
  $F(A) = \big\{y \in Y : (\forall x \in A)\ (x, y) \in R\big\}$
  $G(B) = \big\{x \in X : (\forall y \in B)\ (x, y) \in R\big\}$

Galois connection related to the uniform approximability

define $R_\Phi \subseteq C(X,Y) \times C(X)$, where $\Phi \subseteq C(X,Y)$:
$(f,E) \in R_\Phi$ iff f is uniformly approximable by functions from Φ on E

a Galois connection generated by R_Φ:

$$F_\Phi(\mathcal{E}) = \big\{ f \in C(X,Y) : (\forall E \in \mathcal{E})\ (f,E) \in R_\Phi \big\},$$
$$G_\Phi(\mathcal{F}) = \big\{ E \in C(X) : (\forall f \in \mathcal{F})\ (f,E) \in R_\Phi \big\},$$

where $\mathcal{F} \subseteq C(X,Y)$, $\mathcal{E} \subseteq C(X)$

$\mathcal{L}_\Phi$ – complete lattice obtained from the Galois connection (F_Φ, G_Φ)

Kronecker and Dirichlet sets revisited

Char $= \{\chi_n : n \in \mathbb{Z}\}$, $X = Y = \mathbb{T}$

- $(C(\mathbb{T},\mathbb{T}), \mathcal{K})$ is the top element of the lattice $\mathcal{L}_{\text{Char}}$
- $(\emptyset, \mathcal{P}(\mathbb{T}))$ is the bottom element of $\mathcal{L}_{\text{Char}}$
- $(\text{Char}, \mathcal{D})$ is the only atom in $\mathcal{L}_{\text{Char}}$
- $(\mathcal{C}_k, \mathcal{D}_k) \leq (\mathcal{C}_{k'}, \mathcal{D}_{k'}) \Leftrightarrow k \mid k'$,
  where $\mathcal{C}_k = \{\chi_n + c : n \in \mathbb{Z},\ c \in \mathbb{T},\ kc = 0\}$, $\mathcal{D}_k = G(\mathcal{C}_k)$
- there are pairs between $(\text{Char}, \mathcal{D})$ and $(C(\mathbb{T},\mathbb{T}), \mathcal{K})$ other than $(C_k, \mathcal{D}_k)$

Uniform approximability by constant functions

$\mathsf{Const} = \{f \in C(\mathbb{R}, \mathbb{R}) : f \text{ is constant}\}$, $X = Y = \mathbb{R}$

Fact

$(f, E) \in R_{\mathsf{Const}}$ iff $f \upharpoonright E$ is constant

Question

For what pairs $(\mathcal{F}, \mathcal{E}) \in C(\mathbb{R}, \mathbb{R}) \times C(\mathbb{R})$ one has $(\mathcal{F}, \mathcal{E}) \in \mathcal{L}_{\mathsf{Const}}$?

Theorem

Let $(\mathcal{F}, \mathcal{E}) \in \mathcal{L}_{\mathsf{Const}}$. Then $\mathcal{E}$ is closed under taking

- *closed subsets,*
- *overlapping unions,*
- *Kuratowski lower limits of nets.*

Moreover, $\mathcal{E}$ contains all singletons.

Kuratowski lower limits

Kuratowski lower limit of a sequence of sets:

$$\mathrm{Li}_{n\to\infty} E_n = \Big\{x : \text{there is a sequence } \{x_n\}_{n\in\mathbb{N}} \text{ such that } x_n \in E_n \text{ for every } n \text{ and } x_n \to x\Big\}$$

Kuratowski lower limit of a net:

$$\mathrm{Li}_{p\in P} E_p = \Big\{x : \text{there is a net } \{x_p\}_{p\in P} \text{ such that } x_p \in E_p \text{ for every } p \text{ and } x_p \to x\Big\},$$

where P is a directed partial order

Fact

$\mathrm{Li}_{p\in P} E_p$ *is closed provided that each* E_p *is closed.*

Question

Let $\mathcal{E} \subseteq C(\mathbb{R})$ contains all singletons and is closed under taking closed subsets, overlapping unions, and Kuratowski lower limits of nets.
Does there always exist $\mathcal{F} \subseteq C(\mathbb{R}, \mathbb{R})$ such that $(\mathcal{F}, \mathcal{E}) \in \mathcal{L}_{\mathsf{Const}}$?

Question

Can one replace nets by sequences?
If $\{E_p\}_{p \in P}$ is a net in $C(\mathbb{R})$, can one find a sequence $\{p_n\}_{n \in \mathbb{N}}$ such that $\mathrm{Li}_{p \in P} E_p = \mathrm{Li}_{n \to \infty} E_{p_n}$?

Notation

for $\mathcal{E} \subseteq C(\mathbb{R})$ containing all singletons and is closed under taking closed subsets and overlapping unions,
denote $P_{\mathcal{E}}$ a partition of $\mathbb{R}$ into closed sets such that $\mathcal{E} = \bigcup_{E \in P_{\mathcal{E}}} C(E)$

Quotient spaces

Question

Let $\mathcal{E} \subseteq C(\mathbb{R})$. Does there exists $f \in C(\mathbb{R}, \mathbb{R})$ such that $G_{\mathsf{Const}}(\{f\}) = \mathcal{E}$?

P – a partition of a topological space X
X/P – the quotient space $X/\sim$, where $\sim$ is the equivalence associated with the partition P

Fact

1. $P_{G_{\mathsf{Const}}(\{f\})}$ is the family of all fibers (levels) of f.
2. A partition $P \subseteq C(\mathbb{R})$ is a level set iff the quotient space $\mathbb{R}/P$ is homeomorphic to an interval or a point.

Question

Let $\mathcal{E} \subseteq C(\mathbb{R})$. Does there exists $\mathcal{F} \subseteq C(\mathbb{R}, \mathbb{R})$ such that $G_{\mathsf{Const}}(\mathcal{F}) = \mathcal{E}$?

Fact

$G_{\mathsf{Const}}(\mathcal{F}) = \bigcap_{f \in \mathcal{F}} G_{\mathsf{Const}}(\{f\})$.

Theorem

Let $\mathcal{E} \subseteq C(\mathbb{R})$. The following conditions are equivalent.

1. *There exists $\mathcal{F} \subseteq C(\mathbb{R}, \mathbb{R})$ such that $G_{\mathsf{Const}}(\mathcal{F}) = \mathcal{E}$.*
2. *The quotient space $\mathbb{R}/P_{\mathcal{E}}$ is a connected subspace of cube $\mathbb{R}^X$ for some X.*
3. *The quotient space $\mathbb{R}/P_{\mathcal{E}}$ is Tychonoff.*