Mapping relatively convergent series to divergent series

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Preserving the convergence of series

Theorem (R. Rado)

Let $f : \mathbb{R} \to \mathbb{R}$. The following conditions are equivalent.

1. *f preserves the convergence of series, i.e., for every $\{x_n\}_{n\in\mathbb{N}}$, if $\sum x_n$ converges then $\sum f(x_n)$ converges,*
2. $\exists a \in \mathbb{R}\ \exists\delta > 0\ \forall x \in (-\delta, \delta)\ f(x) = ax$.

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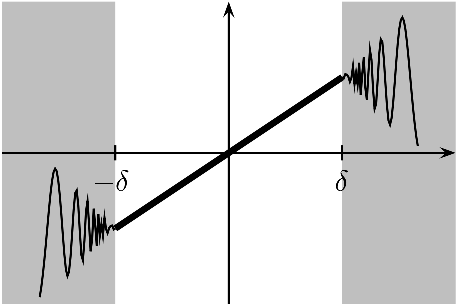

Changing convergence type of a series

Let A, B be families of sequences of real numbers.
Denote $F(A,B)$ the family of all functions $f : \mathbb{R} \to \mathbb{R}$ mapping every sequence $\{x_n\}_{n\in\mathbb{N}} \in A$ to a sequence $\{f(x_n)\}_{n\in\mathbb{N}} \in B$.

Consider the following families:

$$\begin{aligned} C &= \{\{x_n\}_{n\in\mathbb{N}} : \textstyle\sum x_n \text{ converges}\},\\ AC &= \{\{x_n\}_{n\in\mathbb{N}} : \textstyle\sum x_n \text{ absolutely converges}\},\\ RC &= \{\{x_n\}_{n\in\mathbb{N}} : \textstyle\sum x_n \text{ relatively converges}\} = C \setminus AC,\\ D &= \{\{x_n\}_{n\in\mathbb{N}} : \textstyle\sum x_n \text{ diverges}\}. \end{aligned}$$

In Borsík's paper, all families $F(A,B)$ are characterized, for $A \in \{C, AC, RC, D\}$, except $F(D,D)$ and $F(RC,D)$.

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RC &= \Big\{\{x_n\}_{n\in\mathbb{N}} : \sum x_n \text{ relatively converges}\Big\} = C \setminus AC,\\
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Characterizing $F(A,B)$ by a family of closed sets

Fact

Let $A, B \in \{C, AC, RC, D\}$.
If $f \in F(A,B)$ *and* $g \subseteq \mathrm{cl}(f)$ *then* $g \in F(A,B)$.

$g \subseteq \mathrm{cl}(f)$ means that the graph of f is a subset of the topological closure of graph of g

Corollary

Let $A, B \in \{C, AC, RC, D\}$.
There exists $\mathcal{C}$*, a family of closed subsets of* $\mathbb{R}^2$*, such that*
$f \in F(A,B) \iff \exists X \in \mathcal{C}\ f \subseteq X$.

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Functions preserving the divergence of series

Theorem (P.E., Stará Lesná 2008)

$$F(D,D) = \{f : \exists a\ \exists b > 0\ (f \subseteq Y(a,b)\ \vee\ f \subseteq Z(a,b))\},$$

where $Y(a,b) = \{(x,y) \in \mathbb{R}^2 : x = 0\ \vee\ |y| \geq b\ \vee\ |x| \leq ay\}$ and $Z(a,b) = \{(x,y) \in \mathbb{R}^2 : x = 0\ \vee\ |y| \geq b\ \vee\ x = ay\}$.

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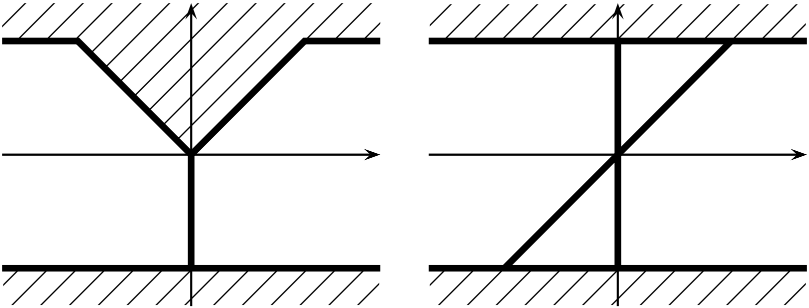
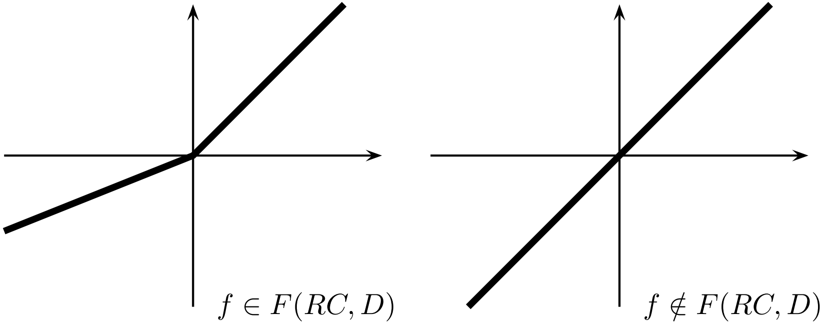


Characterizing family $F(RC, D)$

Let $a, b \in \mathbb{R}$ and $f(x) = \begin{cases} ax & \text{if } x \geq 0, \\ bx & \text{if } x < 0. \end{cases}$

Then $f \in F(RC, D) \iff a \neq b$.



Characterizing family $F(RC, D)$

For $a \in \mathbb{R}$, $b, c > 0$, $d \in \{-1, 1\}$, denote

$$\begin{aligned} K(a,b,c,d) = \{(x,y) \in \mathbb{R}^2 : x = 0 \ \vee \ |x| \geq c \ \vee \ |y| \geq c \vee \\ dy \geq ax + b\,|x|\}, \\ L(c,d) = \{(x,y) \in \mathbb{R}^2 : |y| \geq c \ \vee \ dx \leq 0 \ \vee \ dx \geq c\}. \end{aligned}$$

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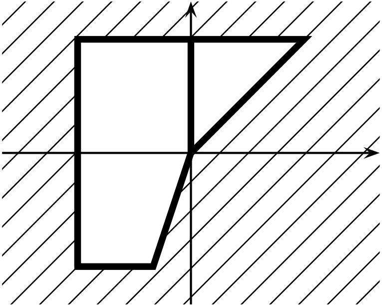

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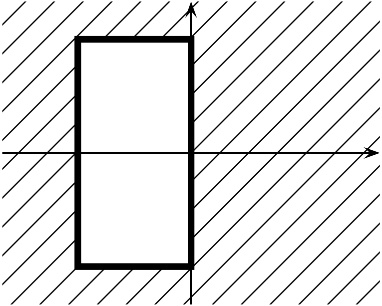

$L(c,d)$

Characterizing family $F(RC, D)$

For $f : \mathbb{R} \to \mathbb{R}$ and $\varepsilon > 0$ denote

$$R_f^-(\varepsilon) = \left\{ \frac{f(x)}{x} : x \in (-\varepsilon, 0) \ \wedge \ |f(x)| < \varepsilon \right\},$$
$$R_f^+(\varepsilon) = \left\{ \frac{f(x)}{x} : x \in (0, \varepsilon) \ \wedge \ |f(x)| < \varepsilon \right\}.$$

Characterizing family $F(RC, D)$

Theorem

Let $f : \mathbb{R} \to \mathbb{R}$. *TFAE:*

1. $f \in F(RC, D)$,
2. $\exists \varepsilon > 0 \; R_f^-(\varepsilon) = \emptyset \;\vee\; R_f^+(\varepsilon) = \emptyset \;\vee$ $\inf R_f^-(\varepsilon) > \sup R_f^+(\varepsilon) \;\vee\; \inf R_f^+(\varepsilon) > \sup R_f^-(\varepsilon)$,
3. $\exists a \; \exists b > 0 \; \exists c > 0 \; \exists d \in \{-1, 1\} \; (f \subseteq K(a, b, c, d) \;\vee$ $f \subseteq L(c, d))$.

Conditions $\inf R_f^-(\varepsilon) > R_f^+(\varepsilon)$, $\inf R_f^+(\varepsilon) > R_f^-(\varepsilon)$ mean that the envelops of sets $R_f^-(\varepsilon)$, $R_f^+(\varepsilon)$ do not overlap.

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Characterizing family $F(RC, D)$

Sketch of the proof:

1. $R_f^-(\varepsilon) = \emptyset \;\Rightarrow\; f \subseteq L(\varepsilon, -1)$, $\quad R_f^+(\varepsilon) = \emptyset \;\Rightarrow\; f \subseteq L(\varepsilon, 1)$.

2. If $\operatorname{dist}(R_f^-(\varepsilon), R_f^+(\varepsilon)) = 0$ then there exists $\{x_i\}_{i=1}^k$ such that $\sum_{i=1}^k |x_i| \geq 1$ and for all j, $\left|\sum_{i=1}^j x_i\right| \leq \varepsilon$ and $\left|\sum_{i=1}^j f(x_i)\right| \leq 2\varepsilon$.

3. If $\operatorname{dist}(R_f^-(\varepsilon), R_f^+(\varepsilon)) > 0$, $\inf R_f^-(\varepsilon) < R_f^+(\varepsilon)$, and $\inf R_f^+(\varepsilon) < R_f^-(\varepsilon)$, then there exists $\{x_i\}_{i=1}^k$ such that $\sum_{i=1}^k |x_i| \geq 1$ and for all j, $\left|\sum_{i=1}^j x_i\right| \leq \varepsilon$ and $\left|\sum_{i=1}^j f(x_i)\right| \leq 3\varepsilon$.

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