# Groups generated by Dirichlet sets

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# Preliminaries

$\mathbb{T} = \mathbb{R}/\mathbb{Z}$ – the unit circle
$\|x\| =$ distance of $x \in \mathbb{R}$ to the nearest integer
since $f(x) = \|x\|$ is periodic, it may be viewed as a function on $\mathbb{T}$
$\varrho(x, y) = \|x - y\|$ – a Polish metric on $\mathbb{T}$

Put $\chi_n(x) = \|nx\|$.
Functions $\chi_n : \mathbb{T} \to \mathbb{T}$, $n \in \mathbb{Z}$, are *characters* of $\mathbb{T}$, i.e., continuous group homomorphisms from $\mathbb{T}$ to $\mathbb{T}$.

# Kronecker's Approximation Theorem

independent $=$ linearly independent over $\mathbb{Q}$

**Theorem (Kronecker's Approximation Theorem)**

*Let $x_1, \ldots, x_k \in \mathbb{R}$ be independent irrational numbers and let $y_1, \ldots, y_k \in \mathbb{R}$ be arbitrary.*
*Then for every $m$ and $\varepsilon > 0$ there exists $n > m$ such that for every $i$, $\|nx_i - y_i\| < \varepsilon$.*

# Dirichlet and Kronecker sets

**Definition**

1. $X \subseteq \mathbb{T}$ is called a *Dirichlet set* if there is an increasing sequence of integers $\{n_k\}_{k\in\mathbb{N}}$ such that $\chi_{n_k} \rightrightarrows 0$ on $X$.
2. $X \subseteq \mathbb{T}$ is called a *Kronecker set* if for every continuous function $f : \mathbb{T} \to \mathbb{T}$ there is an increasing sequence of integers $\{n_k\}_{k\in\mathbb{N}}$ such that $\chi_{n_k} \rightrightarrows f$ on $X$.

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By the original definitions, only closed sets were considered.

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**Fact**

*Every Kronecker set is independent.*

*Proof.* If $x = x_1 + \cdots + x_k$ and $\chi_{n_k} \rightrightarrows f$ on $\{x, x_1, \ldots, x_k\}$ then $f(x) = f(x_1) + \cdots + f(x_k)$.

# Families of trigonometric thin sets

**Definition**

$X \subseteq \mathbb{T}$ is called

1. a *pseudo-Dirichlet set* if there is an increasing sequence of integers $\{n_k\}_{k\in\mathbb{N}}$ such that $\chi_{n_k} \xrightarrow{\mathrm{QN}} 0$ on $X$,
2. an *Arbault set* if there is an increasing sequence of integers $\{n_k\}_{k\in\mathbb{N}}$ such that $\chi_{n_k} \to 0$ on $X$,
3. a *Niemytzki set* if there is a sequence positive reals $\{a_n\}_{n\in\mathbb{N}}$ such that $\sum a_n = \infty$ and $\sum a_n \chi_n < \infty$ for $x \in X$,
4. a *weak Dirichlet set* if $X \subseteq Y$ for some analytic set $Y$ such that for every Borel measure $\mu$ there is an increasing sequence of positive integers $\{n_k\}_{k\in\mathbb{N}}$ satisfying $\int_Y \chi_{n_k} \mathrm{d}\mu \to 0$.

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*Remark.* In a 'standard' definition of weak Dirichlet sets, $Y$ is allowed to be universally measurable.

# Families of trigonometric thin sets

$$\begin{tikzcd}[column sep=small] & & & & & \mathcal{A} \arrow[dr, phantom, sloped, "\subset"] & \\ \mathcal{K} & \subset & \mathcal{D} & \subset & p\mathcal{D} \arrow[ur, phantom, sloped, "\subset"] \arrow[dr, phantom, sloped, "\subset"] & & w\mathcal{D} \\ & & & & & \mathcal{N} \arrow[ur, phantom, sloped, "\subset"] & \end{tikzcd}$$

$\mathcal{K}$, $\mathcal{D}$ have bases consisting of closed subsets of $\mathbb{T}$
$p\mathcal{D}$, $\mathcal{N}$ have bases consisting of $F_\sigma$ subgroups of $\mathbb{T}$
$\mathcal{A}$ has a base consisting of $F_{\sigma\delta}$ subgroups of $\mathbb{T}$
$w\mathcal{D}$ has a base consisting of analytic subgroups of $\mathbb{T}$

# Permitted and additive sets

hereditary $=$ closed under taking subsets

$A + B = \{x + y : x \in A \wedge y \in B\}$

**Definition**

1. Let $\mathcal{F}$ be a hereditary family of subsets of a set $X$.
   A set $A \subseteq X$ is called *$\mathcal{F}$-permitted* if $\forall B \in \mathcal{F}\ A \cup B \in \mathcal{F}$.
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$\mathrm{Perm}(\mathcal{F}) = \{A \subseteq X : A \text{ is } \mathcal{F}\text{-permitted}\}$
$\mathrm{Add}(\mathcal{F}) = \{A \subseteq X : A \text{ is } \mathcal{F}\text{-additive}\}$

**Fact**

1. *If $X \notin \mathcal{F}$ then $\mathrm{Perm}(\mathcal{F})$ is an ideal.*
2. *If $\mathcal{F}$ has a base consisting of subgroups of $G$ then $\mathrm{Perm}(\mathcal{F}) = \mathrm{Add}(\mathcal{F})$.*

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**Theorem (T. Körner, 1974)**

*Let $P \subseteq \mathbb{T}$ be a perfect set. Then there exists a Kronecker set $K \subseteq \mathbb{T}$ such that $P + K = \mathbb{T}$.*

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**Corollary**

*Let $\mathcal{F}$ be a hereditary family generated by proper subgroups of $\mathbb{T}$ containing all Kronecker sets. Then there is no perfect $\mathcal{F}$-permitted set.*