Permitted sets are perfectly meager in transitive sense

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For $x \in \mathbb{R}$, let $\|x\| = \min\{|x-k| : k \in \mathbb{Z}\}$.

Definition. A set $X \subseteq \mathbb{R}$ is called:

- an *N-set* (a *set of absolute convergence*) if there exists a trigonometric series absolutely converging on X which does not absolutely converge everywhere $\iff$ if there exists $\{r_n\}_{n\in\mathbb{N}}$ such that $r_n \geq 0$, $\sum r_n = \infty$, and for all $x \in X$, $\sum r_n \|nx\| < \infty$;
- an N_0*-set* if there exists increasing $\{n_k\}_{k\in\mathbb{N}}$ such that for all $x \in X$, $\sum \|n_k x\| < \infty$;
- an *A-set* (an *Arbault set*) if there exists increasing $\{n_k\}_{k\in\mathbb{N}}$ such that for all $x \in X$, $\lim \|n_k x\| = 0$.

Let $\mathcal{N}$, $\mathcal{N}_0$, $\mathcal{A}$ denote the families of all N-, N_0-, and A-sets, respectively.

Facts. 1. $\mathcal{N}_0 \subseteq \mathcal{N} \cap \mathcal{A}$, $\mathcal{N} \cup \mathcal{A} \subseteq \mathbb{K} \cap \mathbb{L}$;

2. families $\mathcal{N}$, $\mathcal{N}_0$, $\mathcal{A}$ are closed under taking subsets, linear transformations, and generating subgroups of $\langle \mathbb{R}, + \rangle$;
3. $\mathcal{N}$, $\mathcal{N}_0$, $\mathcal{A}$ are not ideals.

Definition. For a family $\mathcal{F}$, we say that a set X is *$\mathcal{F}$-permitted* if $X \cup Y \in \mathcal{F}$ for all $Y \in \mathcal{F}$. We denote Perm($\mathcal{F}$) the family of all $\mathcal{F}$-permitted sets.

Theorem. (Arbault, Erdös, Kholshchevnikova) *Any countable set is permitted for families $\mathcal{N}$, $\mathcal{N}_0$, $\mathcal{A}$.*

Theorem. (Bukovský, Kholshchevnikova, Repický) *Any γ-set is permitted for families $\mathcal{N}$, $\mathcal{N}_0$, $\mathcal{A}$.*

Problem. (Bary) Does there exists a perfect $\mathcal{N}$-permitted set?

Conjecture. (Bukovský) If X is $\mathcal{N}$-, $\mathcal{N}_0$-, or $\mathcal{A}$-permitted then X is perfectly meager, i.e. meager relatively to any perfect subset of $\mathbb{R}$.

Definition. Let $a \in \mathbb{N}^{\mathbb{N}}$ be increasing, $m \in \mathbb{Z}$, $z \in \mathbb{Z}^{\mathbb{N}}$. We say that z is a *good expansion of* m *by* a if

$$m = \sum_{n \in \mathbb{N}} z(n)\, a(n)$$

and for all n,

$$\left| \sum\nolimits_{j<n} z(j)\, a(j) \right| \leq \frac{a(n)}{2}.$$

Facts.

1. For any $m \in \mathbb{Z}$ and any increasing $a \in \mathbb{N}^{\mathbb{N}}$ such that $a(0) = 1$, there exists a good expansion of m by a, possibly more than one.
2. If z is a good expansion then $\mathrm{supp}(z) = \{n : z(n) \neq 0\}$ is finite.
3. If z is a good expansion by a then for all n,

$$|z(n)| \leq \frac{1}{2}\left(1 + \frac{a(n+1)}{a(n)}\right).$$

Notation. For $a \in \mathbb{N}^{\mathbb{N}}$, let

$$A(a) = \left\{x : \lim_{n\to\infty} \|a(n)\, x\| = 0\right\}.$$

Let

$$S = \left\{ a \in \mathbb{N}^{\mathbb{N}} : a \text{ is increasing, } a(0) = 1, \text{ and } \lim_{n\to\infty} \frac{a(n)}{a(n+1)} = 0 \right\}.$$

Fact. Family $\{A(a) : a \in S\}$ is a base of $\mathcal{A}$.

Theorem. *For $a, b \in S$, $k \in \mathbb{N}$, let z_k be a good expansion of $b(k)$ by a. Then $A(a) \subseteq A(b)$ if and only if*

(1) $\forall n\ \exists j\ \forall k \geq j\ z_k(n) = 0$, *and*
(2) $\exists m\ \forall k\ \sum_{n\in\mathbb{N}} |z_k(n)| \leq m$.

Problem. Is there an analogue of the previous theorem for the families $\mathcal{N}$ and $\mathcal{N}_0$?

Notation. For $a, b \in S$, let $a \prec b$ denote that for any sequence $\{z_k\}_{k\in\mathbb{N}}$ of good expansions of $b(k)$'s by a, the conditions (1) and (2) hold true.

Corollary. *A set X is $\mathcal{A}$-permitted if and only if for every $a \in S$ there exists $b \in S$ such that $a \prec b$ and $X \subseteq A(b)$.*

Definition. A subset X of a topological space H is *perfectly meager* if for every perfect set $P \subseteq H$, $X \cap P$ is meager in the relative topology of P.

Theorem. (Bartoszyński) *For $X \subseteq 2^\omega$, X is perfectly meager if and only if for every perfect set P there exists an F_σ-set F such that $X \subseteq F$ and F is meager in P.*

Definition. (Nowik, Weiss) A subset X of a topological group H is *perfectly meager in transitive sense* if for any perfect set $P \subseteq H$ there exists an F_σ-set F such that $X \subseteq F$ and for all $t \in H$, $(F + t) \cap P$ is meager in P.

Theorem. *Every $\mathcal{N}_0$- or $\mathcal{A}$-permitted set is perfectly meager in transitive sense.*

Problem. Is the same true for $\mathcal{N}$-permitted sets?

Problem. Can one prove that the families $\mathrm{Perm}(\mathcal{N})$, $\mathrm{Perm}(\mathcal{N}_0)$, and $\mathrm{Perm}(\mathcal{A})$ are the same?