Permitted sets on Cantor group

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Classical theory of permitted sets

- $(\mathbb{T}, +)$ = circle group = $\mathbb{R}/\mathbb{Z} = [0, 1)$ with addition modulo 1
- $X \subseteq \mathbb{T}$ is Arbault set if there is an increasing sequence $\{n_k\}_{k\in\omega}$ such that $\forall x \in X\ n_k x \to 0$
- $n_k x \to 0$ does not imply $x = 0$ because we calculate modulo 1
- if $x \in \mathbb{T}$ is irrational then $\{nx : n \in \omega\}$ is dense in $\mathbb{T}$
- functions of form $x \mapsto nx$ are characters of $\mathbb{T}$, i.e., continuous group homomorphisms $\chi : \mathbb{T} \to \mathbb{T}$
- there are perfect Arbault sets
- all Arbault sets are meager and have Lebesgue measure zero
- family of Arbault sets is closed under generating of a subgroup of $\mathbb{T}$, is not closed under unions

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- $Y \subseteq \mathbb{T}$ is permitted if $X \cup Y$ is Arbault for every X Arbault
- Arbault, Erdős (1952): every countable set is permitted
- Körner (1972): there is no perfect permitted set
- Bukovský, Kholshchevnikova, Repický (1995): every γ-set is permitted
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Classical theory of permitted sets

Following results are consistent with ZFC $+ \neg$CH:

- $|X| \leq \aleph_1 \Rightarrow X$ is permitted,
- there exists a permitted set of size $\mathfrak{c}$,
- X is permitted $\Rightarrow |X| \leq \aleph_1$.

Open problems:

- does there exist a permitted set of size $\aleph_1$?
- are permitted sets σ-additive?

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Cantor group

- $(2, +) = \{0, 1\}$ with addition modulo 2
- Cantor group $= 2^\omega$ with product topology and $+$ defined coordinatewisely
- a (principal) difference from $(\mathbb{R}, +)$: $\forall x \in 2^\omega \ x + x = 0$
- $X \subseteq 2^\omega$ is Arbault set if there exists a nontrivial sequence $\{\chi_n\}_{n\in\omega}$ of characters of 2^ω such that $\forall x \in X \ \chi_n(x) \to 0$
- what are the characters of 2^ω?
- what is a nontrivial sequence?

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Characters of Cantor group

recall: characters of a topological group G are continuous group homomorphisms $\chi : G \to \mathbb{T}$

- since $x + x = 0$ for all $x \in 2^\omega$, characters of 2^ω can be viewed as continuous group homomorphisms $\chi : 2^\omega \to 2$
- by the continuity of characters and compactness of 2^ω, for every character χ there is $n \in \omega$ such that $\chi(x)$ depends only on $x \restriction n$
- since characters are group homomorphisms, for every character χ there exists a finite set $A \subseteq \omega$ such that

$$\chi(x) = \chi_A(x) = \sum_{n \in A} x(n) \quad \text{(modulo 2)}$$

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Arbault sets in Cantor group

recall: $X \subseteq 2^\omega$ is Arbault set if there exists a nontrivial sequence of characters $\{\chi_n\}_{n\in\omega}$ such that $\forall x \in X\ \chi_n(x) \to 0$

- characters are functions $\chi_A(x) = \sum_{n\in A} x(n)$, where $A \in [\omega]^{<\omega}$
- we do not want nonempty open set to be Arbault

We say that a sequence $\{A_n\}_{n\in\omega}$ of nonempty finite subsets of ω is regular if for every n, $\max A_n < \min A_{n+1}$.

Notions of Arbault sets obtained for the following nontriviality conditions on $\{A_n\}_{n\in\omega}$ are equivalent:

- $\bigcup_{n\in\omega} A_n$ is infinite,
- $\{A_n\}_{n\in\omega}$ is regular.

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Properties of Arbault sets

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- $X \subseteq 2^\omega$ is Arbault set if $X \subseteq \mathrm{Arb}_{\{A_n\}_{n\in\omega}}$ for some regular sequence $\{A_n\}_{n\in\omega}$
- Arbault sets belong to $\mathcal{E}$, σ-ideal generated by closed sets of measure zero
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Example:

$$X = \mathrm{Arb}_{\{2n\}_{n\in\omega}} = \{x \in 2^\omega : \forall^\infty n\ x(2n) = 0\},$$
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The group generated by $X \cup Y$ is 2^ω, hence $X \cup Y$ is not Arbault.

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Properties of permitted sets

$Y \subseteq 2^\omega$ is permitted if $X \cup Y$ is Arbault for every Arbault set Y.

- countable sets are permitted
- permitted sets are closed under generating a subgroup of 2^ω and under finite unions
- every set $X \subseteq 2^\omega$ of size $|X| < \mathfrak{s}$ is permitted

Denote $Perm$ the ideal of permitted sets. Thus $\mathrm{non}(Perm) \geq \mathfrak{s}$.

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Inclusions between Arbault sets

Let $\{A_n\}_{n\in\omega}$, $\{B_n\}_{n\in\omega}$ be regular sequences.

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$X \subseteq 2^\omega$ is permitted if and only if for every regular sequence $\{A_n\}_{n\in\omega}$ there exists $\{B_n\}_{n\in\omega} < \{A_n\}_{n\in\omega}$ such that $\forall x \in X\ \chi_{B_n}(x) \to 0$.

Reg = the family of all regular sequences

Corollary

$\operatorname{add}(Perm) \geq \mathfrak{h}(Reg, <)$.

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Functions of the above form are exactly the continuous group homomorphisms $f : 2^\omega \to 2^\omega$.

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No perfect set is permitted

$X \subseteq 2^\omega$ is independent if for every $n \in \omega$, $\{x_k : k < n\} \subseteq X$, and $\{z_k : k < n\} \subseteq \{0,1\}$, the set $\{i \in \omega : \forall k < n \; x_k(i) = z_k\}$ is infinite.

Theorem (analogue of Kronecker's approximation theorem)

Let $n \in \omega$, $\{x_k : k < n\} \subseteq 2^\omega$ be independent, $\{z_k : k < n\} \subseteq \{0,1\}$. Then there exists $A \in [\omega]^{<\omega}$ such that $\forall k < n \; \chi_A(x_k) = z_k$.

Corollary

Every perfect set $P \subseteq 2^\omega$ there exists a continuous group homomorphism $f : 2^\omega \to 2^\omega$ such that $f[P] = 2^\omega$.

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